MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXV. JUNE 11, 1915. No. 8

Professor H. H. Turner, D.Sc., D.C.L., F.R.S., VICE-PRESIDENT, in the Chair.

Thomas Ralph Merton, B.Sc. (Oxon.), Lecturer in Spectroscopy, King's College, London, 25 Gilbert Street, Grosvenor Square, W.,

was balloted for and duly elected a Fellow of the Society.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended:—

George Ridsdale Goldsbrough, D.Sc., F.L.M.S., 14 Athol Park, Sunderland (proposed by R. A. Sampson).

Professor Solon I. Bailey, Harvard College Observatory, Cambridge, Mass., U.S.A.;

M. Charles Fabry, Professor of Physics in the University, Marseilles; and

Dr. W. H. Wright, Astronomer at the Lick Observatory, Mount Hamilton, California, U.S.A.,

were balloted for and duly elected Associates of the Society.

Forty-two presents were announced as having been received since the last meeting, including, amongst others:

Specola Astronomica Vaticana, Catalogo Astrografico, $+55°$ a $+65°$, calcolate all' Osservatorio di Oxford: vol. i., Decl. $+64°$,

presented by the Observatory; Harlow Shapley, A study of the Orbits of Eclipsing Binaries, presented by the Princeton Observatory; and John Barrett, An enquiry into the origin of the Constellations that compose the Zodiac, Dublin, 1800, presented by E. Walter Maunder.

Stellar Parallaxes determined at the Royal Observatory, Greenwich.

(*Communicated by the Astronomer Royal.*)

1. The first experimental photographs for the determination of stellar parallaxes were taken with the 26-inch photographic refractor at the beginning of 1911. Difficulties were experienced, however, through slight displacements of the crown lens relative to the flint which were not finally overcome till it was mounted in a new cell in July 1912. The adjustment of the lenses was completed by Mr. Davidson after his return from the Solar Eclipse of October 1912, and the observations for parallax were resumed in January 1913. An occulting shutter for equalising the apparent magnitude of the star whose parallax is to be determined with that of the surrounding comparison stars was applied to the telescope whilst the new cell was being constructed and was brought into use at the same time. None of the plates taken before this date are of any value for parallax determinations owing to the above-mentioned want of stability in the mounting of the crown lens.

2. The stars of proper motion exceeding $20''$ a century between Dec. $+64^\circ$ and the pole given in the introduction to the Greenwich Catalogue for 1900, p. 35, were taken as a working list. To this 55 additional stars of large proper motion in the same part of the sky have since been added. Recently a committee of the American Astronomical Society has been formed to co-ordinate measurements of parallax by different observatories, and Greenwich is working in conjunction with it. By this arrangement there will be no unintentional duplication of work, but at the same time it is very desirable that a considerable proportion of stars should be observed at more than one observatory. Professor Schlesinger, the chairman of the committee, has kindly furnished a list of 31 stars out of the 222 of the Greenwich programme which are included in the programme of other observatories.

3. The method proposed by Professor Kapteyn of making exposures at different epochs on the same plate recommended itself in view of the comparatively short focal length of the 26-inch refractor and the ease of making differential measures. It was not, however, attempted to take exposures at three epochs on one plate owing to climatic and other uncertainties. Exposures at two epochs have been taken and the method has been found to work well generally, and there have been comparatively

few failures. It has, however, some disadvantages. It is a drawback to be unable to develop a plate at the time it is taken, as a faulty first exposure may be re-exposed and cause the loss of both exposures. Next, the whole interval over which the observations extend cannot be fully utilised in the determination of proper motion; and any error in this introduces a certain amount of uncertainty into the parallax. On the other hand, the facility of measurement has been found to be a real advantage.

4. The exposures have been made within an hour of the meridian, and re-exposures at the second epoch have always been given within a few minutes of the same hour angle as the first one. Thus the possibility of hour-angle error is completely eliminated. The apparent magnitude of the parallax star has been reduced to be about the $11^{m}\cdot0$ on the photographic scale except in the case of bright stars, where it was not practicable. The occulting shutter is of the rotating sector type driven by clockwork. A series of detachable discs with sectors of various sizes give the required latitude.

5. Eight comparison stars are used. These are selected so as to be as near the reduced magnitude of the parallax star as possible. They are chosen near and symmetrically placed with regard to it, so that their mean position agrees closely with the position of the parallax star. In this way changes of scale and orientation between the two exposures are eliminated. Great importance is attached to obtaining the same magnitude for the mean of the comparison stars and the parallax star in order to secure the elimination of "drift" due to imperfect following from errors in the driving clock, wind on the telescope, or other causes.

The observing programme for each star consists of photographs on six plates taken on twelve nights. Three of these plates are commenced in the evening and re-exposed in the morning some months later. The other three are first exposed in the morning and re-exposed in due time in the evening. At each epoch three exposures are made on each plate. They are arranged in each case at the angles of an equilateral triangle, so that the morning and evening exposures form the alternate angles of a regular hexagon, thus securing the same mean position for the morning and evening images. This symmetrical arrangement is adopted in order to eliminate the possibility of errors due to any local distortion of the film. It is also of value in the measurement, as it minimises errors from runs, screw error, etc.

6. The measurement of the plates is very simple in principle. It is only made in the right ascension co-ordinate. The six images of each star are seen in the microscope together, the micrometer wire is placed on each in succession, and the mean of the three evening images subtracted from that of the three taken in the morning. Approximate co-ordinates of the stars are measured on one plate of the series. From eight equations of the form

$$ax + by + c = x(\mathrm{M}) - x(\mathrm{E})$$

constants a, b, c are deduced for each plate. Applying these to the difference $x(\mathrm{M}) - x(\mathrm{E})$ of the parallax star, its displacement in right ascension between the morning (M) and evening (E) exposures is obtained. If π be the parallax, μ the proper motion, f the parallax factor, and τ the fraction of the year elapsed between the two epochs of exposure, then

$$f \cdot \pi + \tau \cdot \mu = \xi(\mathrm{M}) - \xi(\mathrm{E}).$$

$$f = \mathrm{R}_1 \sin(\odot_1 - \alpha) \cos \mathrm{D}_1 - \mathrm{R}_2 \sin(\odot_2 - \alpha) \cos \mathrm{D}_2,$$

where R_1, $\odot_1$, D_1 are the Sun's radius vector, right ascension and declination at the epoch of the morning observations, and R_2, $\odot_2$, D_2 similar quantities at the epoch of the evening observations, and α is the R.A. of the star. τ is positive if the morning observation is taken after the evening observation, and negative in the converse case. It should be noted that in the latitude of Greenwich the maximum value of f practically attainable is 1·9, and is obtained for stars whose R.A. is about 6^{h}. For stars about 18^{h} which only cross the meridian during darkness in the short summer nights it is difficult to obtain a factor as great as 1·2. Further, the inequality of the coefficient τ in the equations obtained from the two series diminishes the weight of the resulting parallax. It thus happens that the parallax of a star at 18^{h} can only be determined with twice the probable error as one at 6^{h} from the same number of plates.

7. The first star of the list, Lalande 1985, may be given as a specimen of the method of reduction:

(i) 1913 Aug. 20 to 1913 Dec. 31: $1{\cdot}618\pi - {\cdot}363\mu = -{}''{\cdot}078$
(ii) 1914 Aug. 17 to 1915 Jan. 9: $1{\cdot}659\pi - {\cdot}397\mu = -{\cdot}067$
(iii) 1913 Jan. 12 to 1913 Aug. 20: $1{\cdot}624\pi + {\cdot}602\mu = +{\cdot}215$
(iv) 1914 Jan. 5 to 1914 Sept. 3: $1{\cdot}467\pi + {\cdot}660\mu = +{\cdot}165$

Solution of these equations gives

$$\pi = +{}''{\cdot}017 \qquad \mu = +{}''{\cdot}261.$$

Meridian observations give $\mu = +{}''{\cdot}229$. Adopting this value, the above equations lead to the following values of π:—

	Parallax.	Residual.
	$+{}''{\cdot}003$	$-{}''{\cdot}016$
	$+{\cdot}014$	$-{\cdot}005$
	$+{\cdot}047$	$+{\cdot}028$
	$+{\cdot}010$	$-{\cdot}009$
Mean	$+{\cdot}019$	$\pm{\cdot}015$

The probable error is $\dfrac{\pm{}''{\cdot}015 \times {\cdot}87}{\sqrt{3}} = \pm{}''{\cdot}007$.

The values of the parallax are given as determined both ways, and the value of the determined proper motion for comparison with that adopted from meridian observations. The probable error is in each case determined as in the example.

In the table, the visual magnitude is taken from the Bonn Durchmusterung, the photographic magnitude from recent determinations at Greenwich. The type of spectrum has been kindly supplied by Professor Pickering from Miss Cannon's classification. In the column "Proper Motion" the first given for the right ascensions is derived from meridian observations, the second from the parallax plates. Of the two values given for the parallax the first is obtained by assuming the proper motion obtained from meridian catalogues. The mean parallax of the reference stars may be taken as ″·004; this quantity must be added to the parallaxes given to convert them from relative to absolute parallaxes.

	Name and B.D. No.	Mag. Visual Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Dec.	Parallax.	P.E.	No. of Plates.
		m		h m s	″	″	″		
1	Lal. 1985	6·7	F8	1 4 6	+ ·229	+ ·025	+ ·019	± ·007	4
	67°·98	7·0		+ 67° 15′	+ ·261		+ ·017		
2	Lal. 2751	6·7	G5	1 27 20	− ·404	+ ·117	+ ·022	± ·014	5
	68°·113	6·9		+ 68° 26′	− ·398		+ ·021		
3	71°·190	8·9	F8	3 8 17	+ ·219	− ·276	+ ·011	± ·011	6
		9·1		+ 71° 55′	+ ·260		+ 005		
4	68°·278	9·0	Ko	3 38 11	+ ·094	+ ·340	+ ·070	± ·008	5
		9·6		+ 68° 21′	+ ·100		+ ·069		
5	Lal. F. 635	8·5	G5	4 2 5	+ ·055	− ·235	+ ·021	± ·009	6
	75°·166	9·1		+ 76° 2′	+ ·012		+ ·028		
6	66°·343	8·8	G5	4 32 48	+ ·364	+ ·080	+ ·018	± ·006	6
		8·7		+ 66° 32′	+ ·387		+ ·016		
7	Lal. 9428	6·4	F2	5 0 1	000	− ·200	+ ·018	± ·007	5
	64°·500	6·6		+ 64° 48′	− ·015		+ ·020		
8	Groomb. 986	7·3	Go	5 33 16	+ ·178	− ·193	+ ·017	± ·006	5
	74°·257	7·7		+ 74° 34′	+ ·164		+ ·020		
9	Carr. 824	8·8	Go	5 50 13	− ·073	− ·368	− ·005	± ·006	5
	81°·201	9·4		+ 81° 31′	− ·063		− ·007		
10	Carr. 953	8·5	G5	6 41 7	+ ·012	− ·235	+ ·031	± ·008	6
	83°·172	9·1		+ 83° 45′	− ·013		+ ·036		
11	O.A. 7908	8·3	G5	7 23 47	+ ·072	+ ·305	+ ·023	± ·008	4
	73°·380	8·9		+ 73° 29′	+ ·121		+ ·021		

	Name and B.D. No.	Mag. Visual Photog.	Spectral Type.	R.A. 1900·0.	Dec.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
		m		h m s		″	″	″		
12	Pi. VII. 132	6·5	G5	7 39 46	–	·494	+ ·051	+ ·062	± ·007	5
	80°·238	7·0		+	80° 31′	– ·441		+ ·054		
13	Carr. 1196	9·0		8 21 4	–	·071	– ·210	+ ·018	± ·007	5
	85°·127	9·9		+	85° 3′	– ·126		+ ·025		
14	Lal. F. 1384	7·5, 7·6	K2	8 45 59	–	1·343	– ·355	+ ·081	± ·006	5
	71°·482	9·3, 9·6		+	71° 11′	– 1·290		+ ·072		
15	Carr. 1352	9·0		9 14 1	+	·018	– ·442	+ ·055	± ·004	5
	81°·297	10·0		+	81° 1′	+ ·024		+ ·054		
16	O.A. 10038	7·7	F5	9 31 31	–	·242	– ·067	– ·006	± ·006	6
	72°·465	7·7		+	72° 12′	– ·190		– ·010		
17	O.A. 10427	8·2	F8	9 56 3	–	·114	– ·187	+ ·023	± ·005	6
	71°·525	8·6		+	71° 21′	– ·116		+ ·024		
18	O.A. 11040	8·5	Go	10 35 55	+	·191	– ·100	000	± ·006	6
	66°·679	8·6		+	66° 32′	+ ·143		+ ·005		
19	Lal. F_2. 137	8·2	Go	10 56 39	–	·217	– ·050	– ·003	± ·009	5
	81°·359	8·7		+	81° 35′	– ·236		– ·002		
20	Lal. F. 1822	8·7	G5	11 5 10	–	·338	– ·072	+ ·025	± ·006	6
	66°·704	8·6		+	66° 34′	– ·381		+ ·029		
21	Lal. 21927	7·2	F8	11 28 24	–	·055	– ·230	+ ·023	± ·006	8
	66°·724	7·5		+	65° 48′	– ·094		+ ·030		
22	Rad. 2897	8·0	F8	12 36 2	–	·455	+ ·060	+ ·032	± ·012	4
	69°·671	8·0		+	69° 21′	– ·414		+ ·021		
23	84°·284	9·1	Go	12 36 7	–	·201	– ·004	+ ·002	± ·009	6
		9·9		+	84° 11′	– ·185		+ ·001		
24	Groomb. 1923	7·3	Go	12 37 46	–	·232	+ ·009	– ·004	± ·009	6
	84°·286	7·7		+	84° 12′	– ·265		– ·002		
25	78°·439	8·9	G5	12 50 13	–	·246	+ ·032	+ ·010	± ·010	7
		9·9		+	78° 14′	– ·251		+ ·010		
26	O.A. 13407	8·5	G5	13 7 2	–	·730	– ·030	+ ·006	± ·007	6
	68°·714	9·3		+	68° 2′	– ·681		+ ·012		
27	O.A. 13635	9·1	G5	13 21 46	–	·229	+ ·120	+ ·013	± ·010	6
	68°·724	10·0		+	68° 42′	– ·189		+ ·013		
28	O.A. 13803	8·0, 8·5	Go	13 31 12	–	·200	·000	+ ·029	± ·007	4
	68°·730	9·1, 9·5		+	68° 17′	– ·204		+ ·030		

	Name and B.D. No.	Mag. Visual	Mag. Photog.	Spectral Type.	R.A. 1900·0. h	m	s	Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
29	76°·493	m 7·8		G5	13	35	7	+ 76° 25′	+ ″·216	− ″·153	+ ″·018	± ″·006	6
		8·5							+ ·231		+ ·016		
30	Groomb. 2196	5·7		Go	14	57	3	+ 82° 55′	+ ·164	− ·218	+ ·006	± ·007	5
	83°·431	6·1							+ ·204		+ ·003		
31	9 Ursæ Min.	6·7		Go	15	0	19	+ 72° 9′	− ·388	+ ·082	+ ·039	± ·011	5
	72°·664	7·1							− ·313		+ ·037		
32	O.A. 15246	8·9		G5	15	9	37	+ 76° 34′	− ·102	+ ·185	+ ·012	± ·015	6
	76°·552	9·4							− ·144		+ ·012		
33	67°·887	8·7		G5	15	23	3	+ 66° 54′	− ·253	− ·010	+ ·013	± ·009	5
		8·8							− ·293		+ ·016		
34	Lal. F. 2684	8·2		Go	15	31	48	+ 82° 14′	+ ·225	+ ·039	− ·002	± ·014	4
	82°·456	7·9							+ ·221		− ·002		
35	O.A. 15755	9·0		K5	15	49	47	+ 74° 43′	+ ·096	− ·306	+ ·014	± ·013	5
	74°·634	9·8							+ ·077		+ ·016		
36	O.A. 16017	9·0		K	16	8	56	+ 66° 6′	+ ·073	− ·270	+ ·010	± ·007	7
	66°·937	9·6							+ ·075		+ ·010		
37	Lal. 32047	6·3		Ko	17	25	18	+ 67° 23′	− ·547	+ ·012	+ ·063	± ·013	6
	67°·1014	6·8							− ·609		+ ·073		
38	O.A. 17415	9·1		K	17	37	0	+ 68° 26′	− ·353	− 1·241	+ ·269	± ·012	5
	68°·946	9·8							− ·311		+ ·253		
39	35 Draconis	5·1		F5	17	53	56	+ 76° 59′	+ ·048	+ ·239	− ·009	± ·008	6
	76°·667	5·4							+ ·050		− ·009		
40	χ Draconis	3·7		F8	18	22	52	+ 72° 41′	+ ·523	− ·374	+ ·103	± ·011	6
	72°·839	4·4							+ ·486		+ ·120		

The following brief notes may be made on the above results:—

(1) The parallaxes of six of these stars have been determined by other astronomers.

No.	Name.	Greenwich.	Other Determination.	
10.	Car. 953	+ ″·035 ± ″·008	+ ″·001 ± ·023	Groningen
12.	Pi. VII., 132	+ ·066 ± ·007	− ·001 ± ·029	Chase
14.	Lal. F. 1384	+ ·085 ± ·006	+ ·099 ± ·044	Flint
37.	Lal. 32047	+ ·067 ± ·013	+ ·048 ± ·026	Chase
38.	O.A. 17415	+ ·273 ± ·012	+ ·254 ± ·021	Krueger
			+ ·288 ± ·036	Flint
41.	χ Draconis	+ ·107 ± ·011	+ ·119 ± ·038	Chase

No other determinations of the parallaxes of these stars have yet been published.

(2) Three of the stars are known to be binaries.

No. 13. Carrington 1196 is Aitken 1078. The visual magnitudes of the components are $9^{m}\cdot1$ and $9^{m}\cdot2$, and the separation in 1905 was $1''\cdot04$. The two stars are not separated on the photographs, but from the elongation of the images the distance apart appears to be smaller than this now. Professor Aitken has kindly promised to remeasure the star.

No. 14. Lal. F. 1384. This star is Σ1280. The distances and position angles given by the photographs as compared with Dembowski's and Struve's measures are—

	″	°	
1831·9	7·43	33·9	Σ
1867·3	6·43	37·9	De
1914·1	4·63	47·5	Greenwich photographs.

And there are a number of measures between 1867 and 1914.

These show a slight curvature. By measurement of the subtense the combined mass is found to be approximately $\frac{1}{9}\odot \operatorname{cosec}^3\theta$, where θ is the inclination of the radius vector to the line of sight. The mass of the system cannot be less than $\frac{1}{9}$th of the Sun's mass and is very unlikely to be greater than that of the Sun.

No. 28. O.A. 13803. This is Σ1767. Measures of the photographs give—

	″	°
1914·4	4·50	350·8.

The change in position angle since Struve's measures in 1832 is only 3°.

No. 40. χ Draconis is a spectroscopic binary. In Lick Bulletin 181, P is given 281·8 days and $a \sin i = 63 \times 10^6$ km

Nos. 23 and 24. These stars have a common proper motion. They are 157″ distant, and their parallax is only $+''\cdot003 \pm ''\cdot007$.

Nos. 29, 30, 35, and 36 are all moving in approximately the same direction, and opposite to that of the solar motion. They belong to the second stream, but it is possible that they are all moving together, as their parallaxes and proper motions are nearly the same.

No.	R.A.	Dec.	Parallax.	Proper Motion.	Direction.
	h m	° ′	″	″	°
29.	13 35	+76 25	+·022	26	259
30.	14 57	+82 55	+·010	27	262
35.	15 50	+74 43	+·018	32	255
36.	16 9	+66 6	+·014	26	259

It is within the limits of error that these have a mean parallax of +″·016, and a common linear velocity of 74 km./sec. perpendicular to the line of sight.

(3) The parallaxes of the stars which have proper motions of 20″ to 30″ a century are in the mean somewhat smaller than was anticipated. There are 22 stars in the list with proper motions between these limits. Their mean parallax is +″·017 (the mean magnitude being $8^{m}\cdot0$ and mean proper motion 23″·6 a century), as against +″·029 given by Kapteyn's formula.

(4) In the following table the stars are arranged according to their spectral types, and the absolute magnitudes calculated by the formula $M = m + 5 + 5 \log \pi$. The range in spectral type is only from F5 to K5. A clear decrease is seen in the absolute brightness of the stars between F5 and K5.

Type of Spectrum.	Abs. Parallax.	App. Mag.	Abs. Mag.	Type of Spectrum.	Abs. Parallax.	App. Mag.	Abs. Mag.
	″	m	m		″	m	m
F2	+·022	6·4	3·1	G5	+·026	6·7	3·8
F5	− 2	7·7	...	...	+ 25	8·5	5·5
...	− 5	5·1	...	...	+ 22	8·8	5·5
...	+ 4	5·0	...	...	+ 35	8·5	6·2
				...	+ 27	8·3	5·4
F8	+·023	6·7	3·5	...	+ 66	6·5	5·6
...	15	8·9	4·8	...	+ 29	8·7	5·9
...	27	8·2	5·4	...	+ 14	8·9	4·6
...	27	7·2	4·4	...	+ 10	8·5	3·5
...	36	8·0	5·8	...	+ 17	9·1	5·3
...	107	3·7	3·9	...	+ 22	7·8	4·5
				...	+ 16	8·9	4·9
G0	+·021	7·3	3·9	...	+ 17	8·7	4·9
...	− 001	8·8	...				
...	+ 4	8·5	...	K0	+·074	9·0	8·4
...	+ 1	8·2	...	K0	+ 67	6·3	5·4
...	+ 6	9·1	...	K2	+ 85	7·5	7·1
...	0	7·3	...	...	...	7·6	7·2
...	+ 33	8·0	5·6	K5	+ 18	9·0	5·3
...	...	8·5	6·1	K	+ 14	9·0	4·7
...	+ 10	5·7	...	K	+ 273	9·1	11·2
...	+ 43	6·7	4·9	?K	+ 22	9·0	5·7
..	+ 2	8·2	...	?K	+ 59	9·0	7·8
...	+ 57	7·4	6·2				

Where no absolute magnitude is given, an inferior limit may be obtained by assuming the parallax to be not greater than +″·01. On this supposition the absolute magnitude is obtained by sub-

tracting 5 from the apparent magnitude given in column 3. Thus the second star on the list is probably brighter than $2^{m}\cdot7$.

The determinations of parallax were made under Mr. Davidson's direction. In the observations he was assisted by Mr. Edney, Mr. Melotte, and other observers. The measurement of the photographs were generally made by Mr. Davidson and Mr. Martin.

Royal Observatory, Greenwich:
1915 *June* 7.

Observations of the Right Ascensions of Stars in Küstner's Catalogue for 1900 (*Bonn Observations*, vol. x.) *with the Greenwich Altazimuth.*

(*Communicated by the Astronomer Royal.*)

A travelling-wire micrometer was inserted in the Altazimuth in May 1913. As a test of the extent to which personality depending on magnitude was eliminated by its use, and as affording material for investigating such personality in the Transit Circle observers, it was decided to observe the Right Ascensions of Küstner Stars in the zone 24° to 32° North Declination, being the zone of observation of the Greenwich Nine-Year Catalogue, 1910.

The Altazimuth R.A.'s have been compared with Küstner's and the differences grouped by magnitudes from the fourth magnitude downwards. Proper motions have been applied in all cases where they have been previously determined, and in the cases where they have been determined in connection with the Transit Circle Catalogue (these were adopted only when they amounted to $0^{s}\cdot003$ per annum); all remaining stars that showed considerable discordance from Küstner were re-examined for Proper Motion with the aid of the Astronomische Gesellschaft or other Catalogues. About nine-tenths of the stars have been examined for Proper Motion, but the number that have had Proper Motion actually applied is smaller, not more than one-half. There is thus a residual of small Proper Motion left in, and the effect of the solar motion is shown by combining the stars between 6^{h} and 18^{h} R.A., and those between 18^{h} and 6^{h}. The fact that the resulting difference is nearly independent of magnitude is probably accounted for by the fact that a larger proportion of the brighter stars have had their Proper Motions fully determined and eliminated. As the distribution of stars in Right Ascension is tolerably uniform, the effect of the solar motion is eliminated from the Final Means.

Mean Discordance Küstner – Altazimuth arranged in order of Magnitude.

R.A.	m m 4·0 to 4·9.	m m 5·0 to 5·9.	m m 6·0 to 6·9.	m m 7·0 to 7·9.	m m 8·0 to 8·4.	m m 8·5 to 8·9.	m m 9·0 to 9·5.
h h	s	s	s	s	s	s	s
0– 6	+ ·0452	− ·002$_{18}$	·000$_{24}$	− ·015$_{33}$	− ·004$_{43}$	+ ·002$_{58}$	+ ·005$_{33}$
6–12	+ ·0433	+ ·008$_{9}$	− ·014$_{24}$	− ·014$_{30}$	− ·009$_{34}$	·000$_{43}$	− ·002$_{19}$
12–18	+ ·010$_{4}$	·000$_{18}$	+ ·009$_{17}$	+ ·006$_{26}$	+ ·020$_{30}$	+ ·007$_{39}$	+ ·026$_{27}$
18–24	+ ·015$_{6}$	− ·026$_{21}$	− ·013$_{25}$	− ·034$_{28}$	− ·026$_{44}$	− ·029$_{48}$	− ·034$_{36}$
Means	+ ·023$_{15}$	− ·008$_{66}$	− ·006$_{90}$	− ·015$_{117}$	− ·007$_{151}$	− ·005$_{188}$	− ·004$_{115}$
h h	s	s	s	s	s	s	s
6–18	+ ·024$_{7}$	+ ·004$_{17}$	− ·004$_{41}$	− ·005$_{56}$	+ ·005$_{64}$	+ ·003$_{82}$	+ ·016$_{46}$
18– 6	+ ·023$_{8}$	− ·015$_{39}$	− ·007$_{49}$	− ·024$_{61}$	− ·015$_{87}$	− ·012$_{106}$	− ·016$_{69}$

It will be seen that with the exception of the column $4^m{\cdot}0$ to $4^m{\cdot}9$, which depends on a small number of stars, the mean discordance Küstner – Altazimuth is practically constant for all magnitudes, so that the use of the travelling-wire micrometer almost entirely eliminates personality depending on magnitude, at least for the fainter stars.

A Proposal for the Comparison of the Magnitude Scales of the Astrographic Catalogue. Seventh Note. The Vatican Magnitudes. With a Photographic Determination of the Relative Galactic Condensation per Stellar Magnitude. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The last note is in *Monthly Notices*, lxxv. p. 465, where previous references are given; and on p. 480 a diagram is given showing the general disposition of certain presumably obscured patches in the sky. The material of the present paper indicates that of the two portions into which the northern patch divides at the Oxford zones ($+26°$ to $+31°$) the one to the right (8^h–16^h) does not extend so far north as the Vatican zones ($+59°$ to $+65°$): the one to the left reaches these zones in R.A. 20^h–4^h. But it seems unnecessary to give a revision of the diagram until more material is accumulated.

2. Of the Vatican zones, $+64°$ has been printed and $+63°$, $+62°$, $+61°$, $+60°$ are fully reduced ready for press, zone $+63°$ being in type as far as 6^h. Much of the copy for press is at Oxford, where the determination of plate constants is made; and Miss E. F. Bellamy has kindly counted, partly in her leisure time, zones $+64°$, $+62°$, and $+60°$, which cover the sky once from $+59°$ to $+65°$. The intermediate zones $+63°$ and $+61°$ will ultimately be counted, but the information they give is essentially of a duplicating nature, though the diminution of accidental errors by these independent counts is of undoubted value. For the

purpose of a preliminary survey, however, the counts here used are adequate. A summary of the counts is given in Table I. It seems needless to give the details for the separate zones. They have been combined by taking the means per plate for each hour (in which there may be three or four plates according to circumstances) and then the means of these three means again. We might have taken the direct mean of all the plates in all three zones at once, but the procedure adopted allowed of a preliminary comparison, by inspection, of the separate zones, and was chosen for this reason.

TABLE I.

Means of Vatican Counts for Zones $+64°$, $+62°$, *and* $+60°$.

(The unit is one-tenth of a star per plate.)

R.A.	50 and over.	49 to 40.	39 to 30.	29 to 20.	19 to 10.	9 to 0.
0^h	69	93	279	952	3524	4651
1	102	140	371	930	3117	3433
2	72	96	248	606	1957	2352
3	59	82	253	600	1850	1830
4	37	69	200	442	1235	1559
5	36	51	152	429	1512	2226
6	26	54	141	473	1396	2039
7	19	36	135	379	1296	1451
8	17	15	111	263	784	770
9	11	23	97	208	634	571
10	7	23	111	242	650	723
11	14	37	80	191	511	507
12	7	17	63	164	521	621
13	7	12	77	181	476	613
14	12	28	84	170	590	583
15	12	27	74	240	655	687
16	13	39	86	266	857	842
17	13	28	101	263	817	726
18	13	36	153	428	1187	1103
19	18	31	142	521	1738	2382
20	74	84	241	566	1765	1913
21	103	94	263	698	2090	2112
22	63	86	215	628	1728	2007
23	83	113	261	771	2643	2798
Totals	887	1314	3938	10611	33533	38499
N′	887	2201	6139	16750	50283	88782
log N_0'	0·568	0·963	1·408	1·844	2·321	2·568
Mag. C. and M.	8·17	8·99	9·99	11·03	12·30	13·00

3. It has not been considered necessary to exhibit the three zones separately, because it was shown in the case of the Oxford zones (p. 468, *loc. cit.*) that the differences were probably of an accidental character, and the best result was got by combining the zones.

4. At the head of each column are given the limits of diameter of star image between which the counts were made. At the foot are given first the simple totals for each column; then the cumulative totals N'; then the logarithms of these diminished by $\log 240 = 2{\cdot}380$ so as to get results applicable to a single plate. Finally the magnitudes deduced from the table given on p. 145 of this volume (Table II.). The formula adopted in the printed volume of the Vatican results for reducing diameter d to magnitude m is

$$m = \text{const.} - 1{\cdot}02\sqrt{d}.$$

The constant varies from plate to plate, but its mean value is about 14·50. Using this mean value, the comparison would stand thus:—

d	=	50	40	30	20	10	(6)
Mag. by Vat. formula.	=	7·29	8·04	8·91	9·94	11·28	12·00
Mag. C. and M.	=	8·17	8·99	9·99	11·03	12·30	13·00
Diff.	=	0·88	0·95	1·08	1·09	1·02	1·00
Less 1·00	=	−·12	−·05	+·08	+·09	+·02	·00

5. It thus appears that the mean value of the constant 14·50 is about one whole magnitude in error and should be 15·50. The same is true for the Oxford magnitudes, although mention of the fact is omitted by accident from pp. 466 and 467 (*loc. cit.*); for the Bordeaux adopted magnitudes (see p. 473); for the Algiers magnitudes (p. 476). This systematic difference of about 1 magnitude between the scales in current use (deduced in general from visual scales) and that of Chapman and Melotte requires one small correction to which my attention was directed by Mr. R. J. Pocock. The area of a plate is really more than 4 square degrees, owing to the existence of an extra line of 5′ bordering the square of 2″ side. The area is really 169/144 times that assumed, the logarithm of which is ·0695; and this would correct the magnitudes by from 0·1 to 0·2. Of this factor note was taken in the earlier papers of this series (see *M.N.*, lxxii. p. 466), but it has been neglected later, partly because our main concern is with relative and not absolute magnitudes, and partly because the extra area was regarded as a rough compensation for the undoubted falling off of stars near the borders of a plate. This latter consideration can, however, only apply to the faintest stars. The bright stars are not lost or even reduced to a fainter magnitude: on the other hand their images, being diffused (at the border of a plate) credit them as brighter than usual. The point requires detailed consideration, but need not detain us here. It only remains to note that the value of d given

as (6) is not taken from the measures themselves, which are recorded down to $d=4$, but adopted from the remark in § 4 on p. 467 of this volume, that the limit $d=6$ would have suited the faintest stars on the Oxford plates. It seems also to suit the Vatican plates.

TABLE II.—(*Vatican*).

Zones +64°, +62°, +60°. *Log N for Different Magnitudes.*

(The unit is 001 : to each resid. add const. at foot and number in 2nd column.)

Mag.	8·2.	9·0.	10·0.	11·0.	12·3.	13·0.	Gal. Lat. °
0	1941	−076	−132	−076	+001	+050	− 1
1	2007	−069	−156	−198	−189	−189	0
2	2337	−079	−176	−229	−236	−223	+ 2
3	2250	−067	−113	−154	−169	−193	+ 6
4	2049	+010	−022	−077	−124	−113	+10
5	2037	−065	−117	−114	−071	−006	+16
6	1898	+037	−012	+041	+049	+105	+22
7	1748	+025	+073	+105	+150	+160	+29
8	1716	−174	−018	−009	−013	−037	+36
9	1532	+040	+129	+098	+084	+044	+43
10	1301	+203	+378	+380	+340	+331	+49
11	1623	+114	+034	−017	−075	−109	+53
12	1301	+105	+180	+197	+214	+230	+55
13	1342	−031	+180	+198	+162	+180	+54
14	1568	+070	+069	000	+006	−014	+51
15	1568	+059	+028	+078	+063	+049	+46
16	1580	+165	+100	+124	+148	+130	+39
17	1602	+050	+093	+105	+113	+075	+32
18	1602	+120	+245	+296	+284	+251	+25
19	1724	−002	+099	+227	+292	+347	+18
20	2346	−114	−203	−263	−283	−292	+12
21	2491	−163	−287	−329	−352	−375	+ 7
22	2279	−074	−177	−184	−216	−217	+ 4
23	2398	−072	−196	−210	−183	−187	+ 1
Const.		445	936	1379	1850	2090	

6. In Table II. are given the logarithms of the *cumulative* totals formed from Table I. (log N) on the plan adopted in previous papers: viz. to any constituent must be added the corresponding constants in the second column and the bottom row. Thus 1000

times the logarithm of 69 + 93 (the number of stars for which $d > 39$ in 0^h) is

$$1941 + 445 - 76 = 2210.$$

The second column gives the actual logarithms of the first group ($d > 49$) in units of ·001, and the others give the relative differences.

7. Now, it is seen at a glance that the circuit of R.A. is divisible into two main portions: one from 6^h–19^h, in which the residuals are generally positive; and the other (20^h–5^h), in which they are negative. The demarcation between the portions at 19^h–20^h is very abrupt indeed: 19^h is steadily increasing positive, and 20^h is decreasing negative. The other boundary is not so sharp: a recurrence to the separate plates might find a sharp boundary somewhere in 5^h. But at present we are concerned with broad features, and we will let this go. We will further treat as accidental some of the exceptional hours in both groups, *e.g.*, 8^h, 11^h, 14^h, 0^h. Broadly speaking, the increase of faint stars is systematically less in 20^h–5^h than in 6^h–19^h.

8. Now we have remarked in previous papers of this series that the phenomenon at present under discussion has no reference to the Galaxy: but we have never had so clear a demonstration of this fact as in the present instance, where

the portion in which faint stars relatively fail is in and near the Galaxy itself,

as is seen by the galactic latitude (from *Gron. Pub.*, No. 18) given in the last column of Table II. And we can bring out the discontinuity quite clearly by grouping the results in galactic latitude. For simplicity let us, in the first instance, confine attention to the residuals for magnitude 13·0 alone. They fall quite naturally into the following groups, making the middle group the necessary one of 4 terms only.

Table III.

Arrangement of the seventh column of Table II. in Galactic Latitude.

Gal. Lat.	Resid.	Gal. Lat.	Resid.	Gal. Lat.	Resid.	Gal. Lat.	Resid.	Gal. Lat.	Resid.
°		°		°		°		°	
0	−189	+ 6	−193	+18	+347	+32	+075	+49	+331
+1	−187	+ 7	−375	+22	+105	+36	−037	+51	−014
−1	+050	+10	−113	+25	+251	+39	+130	+53	−109
+2	−223	+12	−292	+29	+160	+43	+044	+54	+180
+4	−217	+16	−006			+46	+049	+55	+230
Mean ±2	−153	+10	−196	+24	+216	+39	+052	+52	+124

9. The thick line, which corresponds to the division already mentioned between 20^h–5^h and 6^h–19^h, clearly separates two groups; in each of which there is a decrease of residual with

increasing galactic latitude. The rate per degree as measured by the first pair of groups is $-\cdot043/8 = -\cdot0054$; and from the third and fourth we get $-\cdot164/15 = -\cdot0109$; from the third and fifth $-\cdot092/28 = -\cdot0033$. The fourth and fifth give a result of the opposite sign, $+\cdot072/13 = +\cdot0054$; and accidental error clearly plays a considerable part. But it is difficult to ascribe the difference between groups 2 and 3 to accidental error; it is out of all proportion to the others.

Accepting this jump as a real discontinuity, we may adopt as the galactic latitude change the simple mean of the other four determinations, viz. $-\cdot0036$, and using this we may reduce all five results to galactic latitude zero, obtaining

$-\cdot146$	$-\cdot160$		$+\cdot302$	$+\cdot192$	$+\cdot311$
Mean $-\cdot153$			Mean $+\cdot268$		

10. The total discontinuity is thus $+\cdot421$, very close to the $+\cdot42$ adopted as running through other zones (see *M.N.*, lxxv. p. 480). But we proceed to treat the other columns of Table II. in the same way.

TABLE IV.

Arrangement of Columns of Table II. in Groups of Galactic Latitudes.

Lat. / Mag.	+2°.	+10°.	+24°.	+39°.	+52°.	Rate per Degree.	Calculated by Formula.
9·0	− 74	− 80	+ 45	+28	+ 92	+ ·0013	− ·0004
10·0	−167	−148	+101	+67	+168	+ ·0026	− ·0009
11·0	−179	−188	+167	+79	+152	− ·0005	− ·0014
12·3	−165	−202	+194	+79	+129	− ·0027	− ·0021
13·0	−153	−196	+216	+52	+124	− ·0036	− ·0024

Table IV. shows the mean results. In the seventh column the rate of change per degree is calculated in the crude way just indicated. If we assume the deduced rates to be proportional to the differences of magnitude (shown in the first column) from magnitude 8·2 which was used in Table II. as the standard, weight them according to these differences, and deduce the mean rate per magnitude, we obtain $-\cdot00050$; from which the rates given in the last column were calculated.

The group for $+39°$ does not fall between those for $+24°$ and $+52°$, as we might expect, but it seems doubtful whether this is more than an accident. In the first draft of the discussion, a slight error was made in taking the galactic latitudes, from Mr. Walkey's table (*M.N.*, lxxiv. p. 201) which involves double interpolation. This put 10^h into the group in question and took out 15^h, when the anomaly disappeared. The numbers now given are from Kapteyn's figures.

11. Using these calculated rates, we can correct all the groups to galactic latitude zero as in Table V.

TABLE V.

Results of Table IV. corrected to Galactic Latitude 0°.

Lat. / Mag.	+2°.	+10°.	+24°.	+39°.	+52°.	Obscuration.
9·0	− 73	− 76	+ 55	+ 44	+113	+145
10·0	−163	−139	+122	+102	+215	+297
11·0	−176	−174	+201	+134	+225	+362
12·3	−161	−181	+244	+161	+238	+385
13·0	−148	−172	+274	+146	+249	+382

The obscuration given in the last column is the difference of the simple means of the two sets. The results for +39° are curiously small. This group includes 8^h, which Table II. shows to be exceptional. But, as already remarked, we can only at present study the broad features.

12. The obscuration for 13·0 in Table V. is now shown as +·382, whereas in § 10 it was found to be +·421. The difference is due to the use of a larger value of the galactic condensation in § 9: and it thus appears that the galactic condensation and the obscuration are mixed up in the case of the Vatican results. It was remarked in the last paper (*M.N.*, lxxv. p. 480) that when the "obscuration" was removed, there was revealed a galactic condensation, though in previous papers of this series, and in the memoir of Chapman and Melotte, no sensible condensation could be found. Such figures as those of Tables III. and IV. show how it was hidden. If we make a break between the second and third of these groups, admitting the existence of a discontinuity, numbers *diminish* as we leave the galaxy. But suppose we had ignored this discontinuity? Then we should have obtained on the average a much greater *increase* of the numbers as we leave the galaxy, sufficient to swamp entirely the decrease in other zones not affected so deleteriously as the Vatican zones. The Vatican zones did not enter specially into the work of Chapman and Melotte: but the Greenwich zones are just within those of the Vatican and are doubtless affected in a similar manner. Now the Greenwich zones did quite naturally enter with considerable weight into the discussion of Chapman and Melotte, and are probably responsible for the result obtained—of insensible galactic condensation.

13. We give below the values of the condensation obtained from other zones, separating the "unobscured" from the "obscured" portions. It scarcely needs remark that the hypothesis of "obscuration" is thus given a minimum influence on the determinations.

TABLE VI.

Galactic Condensation

(*i.e.* change of log N per magnitude, per degree of galactic latitude).

Oxford (+26° to +31°.)

Unobscured.			Obscured.		
Hourly Groups.	Gal. Lat.	Coeff.	Hourly Groups.	Gal. Lat.	Coeff.
5	±33°	−·00055	5	+73°	−·00011
5	±11		5	±19	
	Bordeaux (+17°).				
5	±41	−·00215	5	+69	−·00083
5	± 9		5	+29	
	Algiers (+1°).				
5	±51	−·00115	6	±56	−·00137
5	±10		6	±26	
	Perth (−32°).				
7	±39	−·00015	2	−82	−·00114
7	± 9		2	−60	
	Cape (−41°).				
5	±31	−·00390	6	−66	−·00046
5	±10		6	±18	
	Mean	−·00158			−·00078

14. Although there is a large range in the magnitudes of these quantities, they are all consistent in sign. It is doubtful whether anything like a definite value can be assigned as yet. The coefficient seems to increase as we go south, the mean values being

Rome	+62°	−·00050
Oxford	+28	−·00033
Bordeaux	+17	−·00149
Algiers	+ 1	−·00126
Perth	−32	−·00065
Cape	−41	−·00218

but it is doubtful whether the large value −·00390 found for the Cape can be real. The inclusion of some additional hours within the "obscured" boundary would reduce it sensibly. The general mean seems to be about −·0010 to −·0015, and applies to the neighbourhood of magnitude 11·5 (scale of Chapman and Melotte). Kapteyn's visual mean coefficient in this neighbourhood is about −·0016, which is of the same order though rather larger.

15. If the increase southward is real, a further explanation is afforded of the failure to detect the galactic condensation when chiefly northern material is used. The possible difference between the southern and northern hemisphere was mentioned a year ago (see *Observatory*, Feb. 1914, p. 85). See also Mr. Pocock's paper (*M.N.*, lxxv. p. 148) on the Perth Magnitudes.

Note on Errors of Measurement. By H. C. Plummer, M.A.

In *Monthly Notices*, lxxv. p. 483, Miss Winifred Gibson has given a very interesting set of measures made on an astrographic plate. From a discussion of these she concludes that "it seems advisable to take readings in all four orientations of the plate if great accuracy is required in the final measure."

As the practice of confining the measures to two opposite orientations has been very widely adopted, it may be well to show that this is perfectly justified by Miss Gibson's own figures.

From the material presented in her Table I. the following table is constructed :—

Image.	A_1.	B_1.	C_1.	A_2.	B_2.	C_2.
1	+11	+ 6	+ 8	−15	+ 3	+12
2	− 4	+ 3	− 8	−15	+ 1	+14
3	+ 2	−14	− 5	+ 7	+10	−18
4	+ 2	+ 6	− 4	+11	+18	−30
5	+ 2	− 4	+ 1	+14	+17	−32
6	− 1	− 1	− 4	+ 2	− 6	+ 4
7	+ 1	− 4	− 4	−11	− 7	+18
8	− 5	− 6	+ 5	−16	+12	+ 5
9	+ 1	− 7	− 2	− 4	+15	−11
10	− 1	−11	− 3	+ 6	+ 9	−15
Sums (with sign)	+ 8	−32	−16	−21	+72	−53
(without sign)	30	62	44	101	98	159

The unit is ·0001 × ½ cm. The figures under A_1 depend on the measures of the observer A alone. In Miss Gibson's notation they represent $\frac{1}{2}(e_{o,0} + e_{o,180})$ or, as it may be expressed, $\frac{1}{4}(0° + 180° - 90° - 270°)$. The corresponding figures for the observers B and C are given under B_1 and C_1. On the other hand, the figures under A_2, B_2, C_2, give the complete mean for each observer, less the mean for the three observers, or $\frac{1}{3}(2A - B - C)$, $\frac{1}{3}(2B - C - A)$, $\frac{1}{3}(2C - A - B)$. The latter mean is based on no less than 168 bisections executed in four orientations. At the foot of each column is given the sum both with and without regard to sign.

No subtle examination of these figures seems to be required in order to see that the common practice is completely justified. In this, the residuals under A_1, B_1, C_1, which are due simply to the inconsistencies of the measurers in different pairs of orientations, are neglected. But if the method of four orientations were adopted, the residuals under A_2, B_2, C_2, are still left, and these are of an order, roughly speaking, three times greater than those under A_1, B_1, C_1. The measures of B in both respects shows a systematic run which seems slightly anomalous.

It is not intended to dispute Miss Gibson's general deductions, which in themselves are of considerable interest. But they must be qualified by the considerations that in observatory practice it is generally necessary to lay down and adhere to fixed rules, and that a gain in accuracy must be shown to be commensurate with the additional outlay required to secure it. In these respects the rule of two opposite orientations seems practically unaffected by Miss Gibson's results.

Miss Gibson surmises that an experienced observer may find that the same two orientations (not necessarily opposite) give the best results in most cases. This is at least open to doubt. Her material is derived from a single plate, and the systematic deviation of the observer B from C and A suggests strongly some peculiarity affecting all the images measured. It is therefore equally likely that with another plate a different pair of orientations would give the best result. The peculiarity might be always present on plates taken with the same telescope.

Miss Gibson suggests no physical cause to account for the compensations found in orientations which are not symmetrically related. In view of the large discordances between the final results of different observers it is even possible that such compensations bear no direct relation to the accuracy of the measures.

Though Miss Gibson makes no allusion to it, by far the most important point involved in her material is the comparison it affords between the results of different measurers. It is not too much to say that the magnitude of the differences from the total mean is truly astonishing. Among no more than ten stars measured in the most elaborate way the observer C in two cases differs by more than 1″ from A and B in estimating the centre of the image. Too little attention has been given to this source of uncertainty, which, quite apart from the actual errors of the *plate*, of itself sets a practical limit to the labour which an individual observer is justified in giving to his measures. A similar point arose in the discussion of some material provided by two French measurers (*Monthly Notices*, lxii. p. 514). The individual discordances in that case seem to have been considerably smaller, the greatest difference among six exposures of eight stars being 0″·5. But, on the other hand, another point there came to light. For in five stars out of forty-three (12 per cent.) the personal discordances proved systematic for all the six exposures, leaving mean differences

of 0″·1 or more, and in one case 0″·25. This example shows that the measurer is guarded against the influence of his own idiosyncrasies neither by multiplying the orientations in which the plate is placed, nor even by multiplying the exposures on a single plate.

The serious discordances between the three observers lead one to hope that the material is not typical. Among the individual measures may be quoted:

Star 4	A 180° ·8647	C 180° ·8570	Diff. ·0077.
Star 5	B 270° ·2753	C 180° ·2668	Diff. ·0085.

Each of these measures is based on fourteen settings in one orientation by observers apparently free from personality in the ordinary sense. It is difficult to understand such results as these. The most charitable view is to assume that the images were so abnormally bad that they should not have been included in so restricted a sample.

The Solar Rotation in June 1911, *from Spectrographic Observations.* By J. B. Hubrecht, M.A.

The following is an abstract from a memoir of the same title, published in the *Annals of the Solar Physics Observatory*, vol. iii. part 1, and contains an account of the principal results obtained in the investigation, as well as of their tentative interpretations.

Reference may also be made to the author's earlier communication to the Society, *Monthly Notices*, vol. lxxiii. pp. 5–27, 1912. Two sets of observations were on that occasion dealt with: one concerning only the solar rotation at the equator and made at epochs extending over a considerable range of time (Dec. 1910–Oct. 1911), the other, made in the first half of June 1911, giving comparisons of radial velocities at points all round the Sun. The latter set consisted of one series of plates out of four which had been taken. The other three series have since been measured by Mr. N. Tunstall, a student of the Victoria University of Manchester. It appeared at once that the consistency of his measures was a good deal greater than that of the author's original measurements of the first series, and accordingly this first series was also re-measured by Mr. Tunstall. The results relating to all four series may thus be regarded as affording homogeneous material for discussion.

It is well further to recall the special feature in which the

investigation differs from those made on the same subject by other observers. Throughout the series of observations here discussed, plates were taken comparing points on the Sun's limb 90° apart from one another. With instrumental conditions otherwise unchanged four observations were made: one comparing the spectrum at a point 1 (fig. 1) with that at point 2, the next that at 2 with that at 3, the next 3 with 4, and finally 4 with 1. After suitably altering the instrumental conditions, a second set of such observations was made with a different starting-point. A complete "series" of observations—of which each observation comprised two plates, taken in two adjoining regions of the solar spectrum—consists of six of these sets of, accordingly, eight plates, starting from 0° E., 15°, 30°, 45°, 60°, and 75° S.E. respectively. Such a

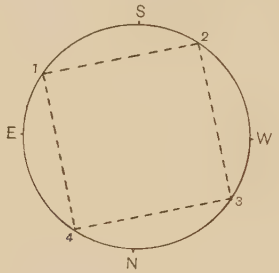


Fig. 1.—Typical group of observations.

series contains therefore observations taken at intervals of 15° round the Sun's limb. The four series discussed contain altogether 192 plates, which were taken, respectively, within the following outside dates:—

Series I., June 1–5, 1911.	Series III., June 5– 8, 1911.
Series II., June 1–7, 1911.	Series IV., June 9–15, 1911.

As to the measurements it must be noted that Mr. Tunstall employed the ordinary eyepiece throughout. The author's method of employing a cylindrical lens, as in the measures discussed in the earlier paper already referred to, and as described in detail at the time in *M.N.*, lxxiii. pp. 28–31, was tried by him but rejected in favour of the ordinary method. For an account of the measuring instrument reference may also be made to Professor Newall's description of a similar instrument given in *M.N.*, lxv. p. 648.

Fifteen lines were measured in each of the two spectral regions, *i.e.* on each plate. Table I. gives the wave-lengths, origins, and intensities of the lines, while it also shows which of them belong to the class of enhanced lines. No elements of atomic weight greater than 59 have lines in this part of the spectrum, while on the other hand the exposures were such that the hydrogen line Hγ at λ 4341 was unsuitable for measurement.

TABLE I.

Selected Lines.

Region λ 4300 – λ 4350.				Region λ 4350 – λ 4400.			
Wave-length.	Element.	Intensity.	Remarks.	Wave-length.	Element.	Intensity.	Remarks.
4299·149	Ca	3	...	4351·000	Ti	1	...
4302·085	Ti	2	Enhanced	4352·908	Fe	4	...
4314·248	Sc	3	...	4358·670	Fe	2	...
4315·138	Ti	3	Enhanced	4366·061	Fe	2	...
4320·907	Sc	3	...	4371·442	Cr	2	...
4326·923	Fe	2	...	4373·415	Cr	1	...
4331·811	Ni	2	...	4373·727	Fe	2	...
4337·725	Cr	3	...	4376·107	Fe	6	Chromospheric
4338·084	Ti	4	Enhanced	4379·396	V	4	...
4339·882	Cr	3	...	4385·144	Cr	2	...
4343·861	Fe	2	...	4388·571	Fe	3	...
4344·670	Cr	4	...	4390·149	V	2	...
4346·725	Fe	2	...	4395·201	Ti	3	Enhanced
4347·403	Fe	1	...	4396·008	Ti	1	...
4349·107	Fe	2	...	4400·555	Sc	3	...

Results from Different Wave-lengths.

It will be convenient first to give an account of the results when viewed with regard to concordance between individual lines. By such a scrutiny an unexpected dependence of velocity on wave-length has been brought to light, which will have to be dealt with before we can proceed to the consideration of the mean results, plate by plate, *i.e.* round the Sun's limb.

For an averaging of the plates from the point of line-to-line effect it is best to group the material in such a way that velocity-differences of about the same amount are brought together. Table II. shows the manner of this grouping. There are five combinations of latitudes which give velocity-differences approximating to $-2{\cdot}2$ km./sec., and also five which give about $+2{\cdot}3$ km./sec. We need for the purposes of line-to-line comparison regard only the numerical values, disregarding the sign. Hence in the next three columns in each case two positive and two negative velocity-differences can be grouped together, just as we might have taken together the ten observations of the first two columns. The observations of the last column will give no information with regard to line-to-line effect, but may afford a useful check in the matter of accidental errors.

TABLE II.

Arrangement of Material in Groups with approximately the same Numerical Velocity-difference.

−2·2 km./sec.		+2·3 km./sec.		±1·9 km./sec.		±1·4 km./sec.		±0·8 km./sec.		0·0 km./sec.	
ϕ_1.	ϕ_2.	ϕ_1.	ϕ_2.	ϕ_1.	ϕ_2.	ϕ_1.	ϕ_2.	ϕ_1.	ϕ_2.	ϕ_1.	ϕ_2.
15° S.E.	75° S.W.	15° N.W.	75° N.E.	0° E.	90° S.	75° S.W.	15° N.W.	60° S.W.	30° N.W.	45° S.W.	45° N.W.
30° S.E.	60° S.W.	30° N.W.	60° N.E.	90° S.	0° W.	15° S.W.	75° N.W.	30° S.W.	60° N.W.	45° N.E.	45° S.E.
45° S.E.	45° S.W.	45° N.W.	45° N.E.	0° W.	90° N.	75° N.E.	15° S.E.	60° N.E.	30° S.E.		
60° S.E.	30° S.W.	60° N.W.	30° N.E.	90° N.	0° E.	15° N.E.	75° S.E.	30° N.E.	60° S.E.		
75° S.E.	15° S.W.	75° N.W.	15° N.E.								

Taking means in this way, results are obtained as shown graphically in fig. 2. The abscissæ are there the wave-lengths; the mean velocity-differences per line per group from Table II. determine the ordinates. Each division of the figure corresponds to a column in the table. The zero points of the axis of ordinates are different in each case, being always, except for the bottom figure, a long way off the page. The scale is otherwise the same for all.

It is at once apparent that some systematic effect is present. The dots do not cluster round horizontal lines as might be expected, but round lines of downward inclination towards the right (greater wave-lengths). There is no evidence of discontinuity at the central points which separate the two spectral regions (denoted A and B in the figure). The inclination is most marked in the two upper divisions of the figure, less so in the three following ones, and practically vanishes in the bottom division. The amount of the inclination varies approximately with the amount of the numerical mean velocity-difference for the groups of plates. If, as seems a sufficient approximation, we suppose the dots to cluster round straight lines, we find for the diminution in $v_{\phi_1} - v_{\phi_2}$ from λ 4300 to λ 4400, indicated by the inclination of these lines, values as in Table III. The constant ratio is fairly evident.

TABLE III.

Diminution along the Plate as determined from the Six Divisions of Fig. 2.

Mean $v_{\phi_1} - v_{\phi_2}$ (numerical)	2·2	2·3	1·9	1·4	0·8	0·0
Diminution from λ 4300 to λ 4400	0·060	0·066	0·060	0·048	0·030	0·006

Both lines of the table being expressed in km./sec., we get for the amount of the diminution for the 100 λ involved 3 per cent.

An exhaustive search into the probable cause of this result is given in § 9 of the original memoir. The method of reduction, the instrumental equipment, the method of observation—including the effect of regional differences on the Sun,—and the measurement are there all reviewed, and the conclusion is reached that it is unlikely that the source of the anomaly is to be found in any of these.

It would appear that in the Sun itself some physical cause is active in producing an effect (at all points of the limb) which

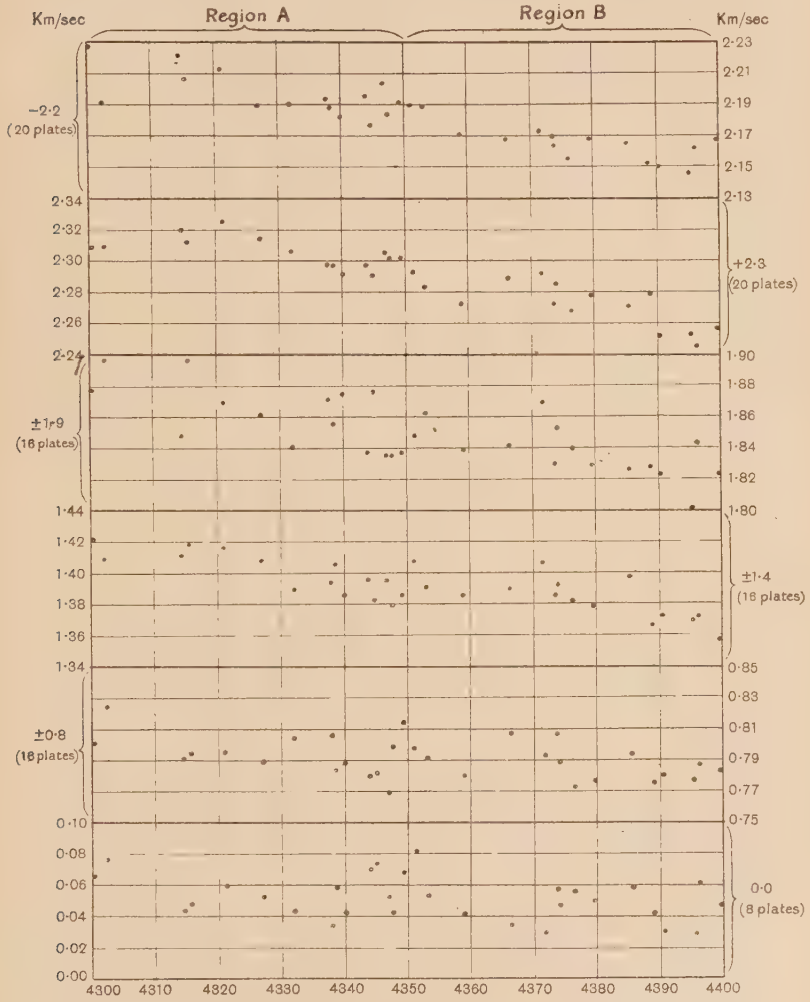


FIG. 2.—Mean velocity-differences for individual lines for material grouped, as in Table II., according to numerical value.

can, within the limits of spectrum here examined, be expressed by the formula

$$v_\lambda = v_{\lambda_0} \cdot \frac{\lambda_1 - \lambda}{\lambda_1 - \lambda_0} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\lambda_0 = 4350$, the central wave-length of our region and λ_1 another constant. Light of different wave-lengths reaching the

observer along a given ray may originate in different levels of the Sun. This in itself would not give rise to a varying velocity in the line of sight. *A varying velocity in the line of sight would result if the rays were curved while passing through these levels.* The curvature, again, would be different for different wave-lengths on account of the change of refraction with the wave-length. The latter cause would generally make the velocities smaller for the more refrangible rays, the former cause larger. If the former preponderates, an effect of the kind observed would be the result. It appears at any rate as if the phenomenon here discussed, and apparent in fig. 2, may have an important bearing on the question of the true place of origin of the absorption lines in the spectrum of the Sun's limb.

With regard to the special behaviour of *individual* lines, if we assume formula (1) to represent the average state of things, and if then we determine the remaining residuals for each line with this average state (putting the proper value for λ_1 in the formula so as to get the 3 per cent. ratio of diminution), it is found that no systematic arrangement remains. Expressed in percentages the largest of the residuals then amounts to 0·8 per cent. When grouped according to origin or intensity no evidence of special behaviour appears. We are thus justified in regarding the residuals as wholly due to accidental error. The mean error for one line amounts under those circumstances to 0·3 per cent.

Velocity-differences at Different Latitudes.

The conclusions drawn as a result of the line-to-line comparison need not deter us from attaching full weight to most of the results which can be obtained from a review, plate-to-plate, of the same material. The mean values per plate must be affected by the influence which, as we saw, has caused the velocity-differences to increase with increasing wave-lengths. But even so, we found this diminution to be proportionate in amount to the mean velocity-difference on the plate, and the presumption is therefore very great that these means themselves, if they have been affected at all, are changed also in the same proportion—that is, by a factor constant for all. Although it is unfortunate that, on this score, the results obtained in the present investigation will carry little weight as to the *absolute* value of the Sun's rotational velocity, yet the comparison of velocities obtained at the various points all round the Sun's limb may be fully carried through with the conviction that any *relative* results, such as have regard to the latitude law, the behaviour in different solar hemispheres, or the general concordance in the velocities for the epoch of observation, should carry the full weight to which their internal agreement entitles them.

Taking means per observation (two plates, taken in the adjoining spectral regions) and then combining the four observations taken under otherwise similar circumstances in the four

series, and so constructing a "Normal Series," the velocity-differences given in Table IV. were obtained.

TABLE IV.

Velocity-differences of Normal Series.

ϕ_1.	ϕ_2.	$v_{\phi_1} - v_{\phi_2}$.	$v_{\phi_1} - v_{\phi_2}$.	ϕ_1.	ϕ_2.
45° S.E.	45° S.W.	−2·262		45° S.E.	45° S.W.
30° S.E.	60° S.W.	−2·195	−2·198	60° S.E.	30° S.W.
15° S.E.	75° S.W.	−2·125	−2·125	75° S.E.	15° S.W.
0° E.	90° S.	−1·854	−1·842	90° S.	0° W.
15° N.E.	75° S.E.	−1·441	−1·447	75° S.W.	15° N.W.
30° N.E.	60° S.E.	−0·847	−0·849	60° S.W.	30° N.W.
45° N.E.	45° S.E.	−0·044	−0·060	45° S.W.	45° N.W.
60° N.E.	30° S.E.	+0·738	+0·732	30° S.W.	60° N.W.
75° N.E.	15° S.E.	+1·341	+1·336	15° S.W.	75° N.W.
90° N.	0° E.	+1·849	+1·852	0° W.	90° N.
75° N.W.	15° N.E.	+2·231	+2·235	15° N.W.	75° N.E.
60° N.W.	30° N.E.	+2·304	+2·307	30° N.W.	60° N.E.
45° N.W.	45° N.E.	+2·369		45° N.W.	45° N.E.

The average probable error of the figures in this table, if determined from the agreement between the four series, amounts to less than ±0·004 km./sec. It may be mentioned here that the average probable error of Mr. Tunstall's measurements per line per exposure is ±0·023 km./sec., or ±0·0004 mm. before reduction. These errors compare favourably with those of other measurers, especially if we consider that they relate to *velocity-differences*, whereas in earlier investigations the probable errors published always relate to velocities, the factor $\frac{1}{2}$ having been introduced in the velocity-coefficient.

The advantages pertaining to the special method of observation employed can be fully realised in a consideration of Table IV.

We can at once conclude that for the epoch of observation *close symmetry existed with respect to the solar axis*, as denoted by the agreement of the corresponding values in the two columns. It may be added that this holds good if the velocity-differences of each of the four series were separately examined.*

As to symmetry with respect to the equator, if this existed it is clear that the values of the table ought to be grouped symmetrically, except in regard to sign, with respect to the middle horizontal line of the table. The nature of the actually occurring *asymmetry* is quite similar to that which would occur if all the velocities in the

* In the original observations the symmetry is not immediately apparent. There is a systematic asymmetry, which allows us to conclude that either an error has been made in the assumed position of the axis, or that a real change has occurred, amounting to 0° 30′, and that close symmetry exists with respect to this changed or corrected axis. See the original memoir, pp. 57–60.

Northern hemisphere were numerically larger than the corresponding velocities in the Southern. Owing to the method of observation we are, however, unable to make a positive statement of this kind which is at the same time quite general. But we can go a little further in disentangling the velocities in the two hemispheres. Assuming the symmetry with respect to the axis to be complete, as we have a good reason for doing, we can, by first combining the top and bottom velocity-differences of the table, and then by combining the two given in the middle horizontal line, deduce two independent determinations of

$$v_{45^\circ \text{N.}} - v_{45^\circ \text{S.}},$$

where, owing to our assumption, no further designation, in the suffixes, of Eastern or Western half of the sun would have any significance, and so is left out. By making other suitable combinations of velocity-differences given in the table, the assumption of symmetry with respect to the axis makes it possible to find two independent values for

$$(v_{15^\circ \text{N.}} - v_{15^\circ \text{S.}}) + (v_{75^\circ \text{N.}} - v_{75^\circ \text{S.}})$$

and for

$$(v_{30^\circ \text{N.}} - v_{30^\circ \text{S.}}) + (v_{60^\circ \text{N.}} - v_{60^\circ \text{S.}}).$$

The resulting figures for each of the four series of observations, as well as their final means, are given in Table V. The last column is obtained by adding together the figures of the three other columns. It obviously allows the comparison to be made between the sum of the velocities in one hemisphere and that in the other. The column gives therefore the values, as found in the present investigation, for the quantity E of the author's earlier publication.* The marked consistency of the figures in Table V. together with the smallness of the probable error of Mr. Tunstall's measurements as compared with that of the author's, justify the course of allowing their full weight to the present results and of disregarding altogether the conclusions drawn in the 1912 publication.

TABLE V.

Comparison of Northern with Southern Hemisphere.

	$(v_{15^\circ \text{N.}} - v_{15^\circ \text{S.}}) + (v_{75^\circ \text{N.}} - v_{75^\circ \text{S.}})$	$(v_{30^\circ \text{N.}} - v_{30^\circ \text{S.}}) + (v_{60^\circ \text{N.}} - v_{60^\circ \text{S.}})$	$(v_{45^\circ \text{N.}} - v_{45^\circ \text{S.}})$	$(v_{15^\circ \text{N.}} + \ldots + v_{75^\circ \text{N.}}) - (v_{15^\circ \text{S.}} + \ldots + v_{75^\circ \text{S.}})$
Series I.	+0·105	+0·107	+0·052	+0·264
	+0·105	+0·103	+0·044	+0·252
Series II.	+0·122	+0·118	+0·045	+0·285
	+0·116	+0·110	+0·042	+0·268
Series III.	+0·103	+0·098	+0·054	+0·255
	+0·103	+0·108	+0·067	+0·278
Series IV.	+0·102	+0·113	+0·062	+0·277
	+0·097	+0·131	+0·055	+0·283
Final means	+0·107	+0·111	+0·052	+0·270

* *M.N.*, lxxiii. p. 24, 1912.

That the values in Table V. are all positive does not necessarily mean that *all* velocities in the Northern hemisphere are larger than the corresponding ones in the Southern. In either or both of the two first columns one of the two terms making up the total might be zero or even negative. The observations in themselves afford no evidence in this respect, owing to the method by which they were made. Some hypotheses may seem more likely than others; we shall have occasion to refer to them below.

It is impossible to obtain, without further assumptions, the actual velocities at the different points from the velocity-differences of Table IV. (p. 617). The adopted method of observation is, however, in no way inferior in this respect to the usual method. From the available material it *is* possible to infer, by simple additions from each four velocity-differences which together complete a closed circle round the limb, four velocity-differences between points at the ends of two diameters, perpendicular to one another, and so to convert the material into such as is usually obtained. Then, making the assumption of symmetry with respect to the axis, from a corresponding but different set of four observations we can deduce four velocity-differences at the ends of diameters symmetrically situated, so that, *in toto*, we have four determinations of $v_{\phi N.} + v_{\phi S.}$ in each case, except in the case of the equator, where we have only two. This has been done for each of the four series; the results are given in Table VI. The figures for each series are the means from the four (or two) independent determinations just mentioned.

Table VI.

Mean $v_{\phi N.} + v_{\phi S.}$ for the Four Original Series.

	$2v_0$.	$v_{15^\circ N.} + v_{15^\circ S.}$	$v_{30^\circ N.} + v_{30^\circ S.}$	$v_{45^\circ N.} + v_{45^\circ S.}$	$v_{60^\circ N.} + v_{60^\circ S.}$	$v_{75^\circ N.} + v_{75^\circ S.}$
Series I.	3·718	3·573	3·047	2·316	1·460	0·788
Series II.	3·701	3·572	3·031	2·322	1·451	0·790
Series III.	3·708	3·570	3·047	2·320	1·464	0·785
Series IV.	3·668	3·565	3·046	2·305	1·465	0·789
Means	3·699	3·570	3·043	2·313	1·460	0·788

It is important, especially with a view to the discussion about the latitude law, to note the close agreement between the velocity-sums as found from the four series. This agreement shows that within the interval covered by the investigation neither did any change occur in the general rate of rotation, nor did any local effects appear. An exception as to the latter conclusion must perhaps be made for the equatorial velocity of Series IV. It is smaller than the mean to a greater extent than is warranted by the general agreement of the table. The observations from which the velocity-sum in question was obtained were all made on June 8, 1911. Observations at other latitudes were made on the same day which do not give diminished velocities for those latitudes. We

probably have to do, in this case, with the effect of a local disturbance, either at 0° E., or at 0° W., or else with a disturbance affecting the velocity all round the equator, and so appearing at both these points.

If these *mean* velocities from the two hemispheres ranged themselves in the usual way according to Faye's law, it should be

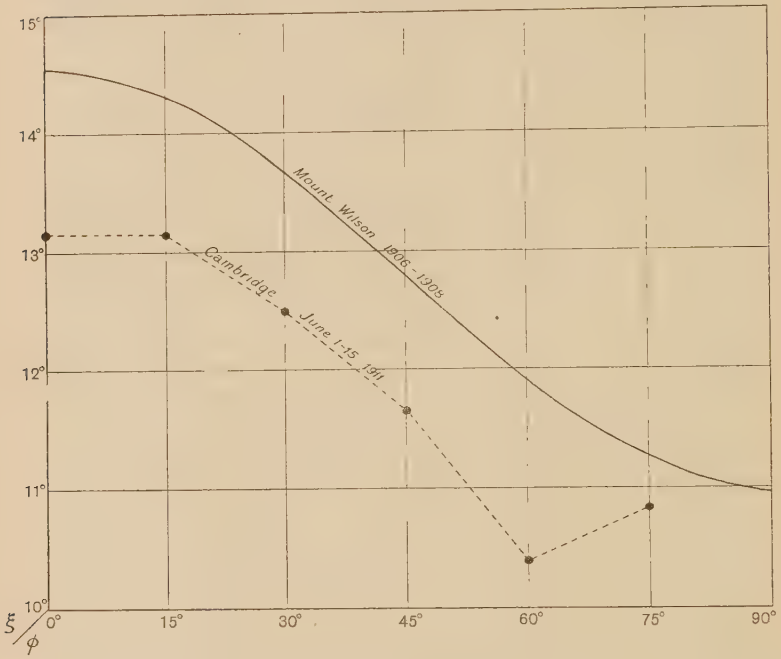


FIG. 3.—Mean angular velocities from four series, reduced to one quadrant, under assumptions of symmetry with respect to axis, and, for the purpose of this reduction, also with respect to equator. Comparison with typical earlier results. The size of the dots is a measure of the range of variation found between the results of the individual four series.

possible to determine from the bottom line of figures of Table VI. the two constants a and b of a formula

$$v_{\phi N.} + v_{\phi S.} = 2(a - b \sin^2 \phi) \cos \phi \quad . \quad . \quad . \quad (2)$$

expressing this law. In attempting to do so it is at once apparent that the figures cannot possibly be represented by a law of this sort. The best way to demonstrate this is by showing the graph which represents the *angular* velocities corresponding to the mean velocity-sums of the table, as well as that given by a representative formula of the above kind. This is done in fig. 3, where the continuous line is given by

$$\xi = 14^{\circ}{\cdot}54 - 3^{\circ}{\cdot}50 \sin^2 \phi,$$

and is taken from the Mount Wilson observations made 1906–1908,* while the broken curve connects points determined directly from the figures in Table VI. by reducing to angular velocities. The diameters of the dots representing the observations correspond in scale to the mean range in angular velocities found when comparing the four series. Table VII. gives the angular velocities for the four series. In both curve and table we have for the moment neglected the difference between the Northern and Southern hemispheres and are dealing with average results from the two.

TABLE VII.

Angular Velocities, averaged over the two Hemispheres, computed from the Figures in Table VI.

	0°.	15°.	30°.	45°.	60°.	75°.
Series I.	13°·22	13°·16	12°·51	11°·65	10°·39	10°·82
Series II.	13·16	13·15	12·45	11·68	10·31	10·86
Series III.	13·19	13·15	12·52	11·67	10·41	10·79
Series IV.	13·05	13·13	12·51	11·59	10·42	10·84
Means	13·16	13·15	12·50	11·65	10·38	10·83

From fig. 3 we see that the two curves differ entirely in character. It is abundantly evident that the Faye formula cannot be applied to the present observations. The author was in fact unable to discover any simple formula with few constants to which the observations approximate.

Equally unsuccessful was the search, if the difference between the two hemispheres had also to be accounted for. We have seen that the observations do not afford means of definitively determining the velocities at the various latitudes in the Northern and Southern hemispheres, even under the adopted assumption of symmetry with respect to the axis. Only for latitudes 45°, for which both $v_{45\,N.} + v_{45\,S.}$ and $v_{45\,N.} - v_{45\,S.}$ are known, can we assign definitive velocities, viz.,

$$v_{45°\,N.} = 1{\cdot}183 \text{ km./sec.}$$
$$v_{45°\,S.} = 1{\cdot}131 \text{ km./sec.}$$

The equatorial velocity can also be directly deduced from the first column of Table VI., viz.,

$$v_0 = 1{\cdot}850 \text{ km./sec.}$$

For the other latitudes 15° N., 15° S., 75° N., 75° S., and 30° N., 30° S., 60° N., 60° S., two sums of velocity-differences and four velocity-sums are given in the bottom lines of Tables V. and VI. If for the velocities at two of these latitudes (one in each group of four) arbitrary values are assumed, those at the remaining six latitudes follow from the observations. For five pairs of such

* *Carnegie Institution of Washington, Publ.* 138, 1911.

arbitrary values, linear velocities have been deduced, and the corresponding angular velocities are plotted in the graphs given in fig. 4, which accordingly represent five alternative interpretations of the observations.

Certain features are bound to appear in all of them. The three angular velocities at 15° S., 0°, and 15° N. must lie on a straight line, whilst towards at least one of the poles there must be an increase in angular velocity over that at a corresponding neighbouring latitude 75°—requirements which are obvious when

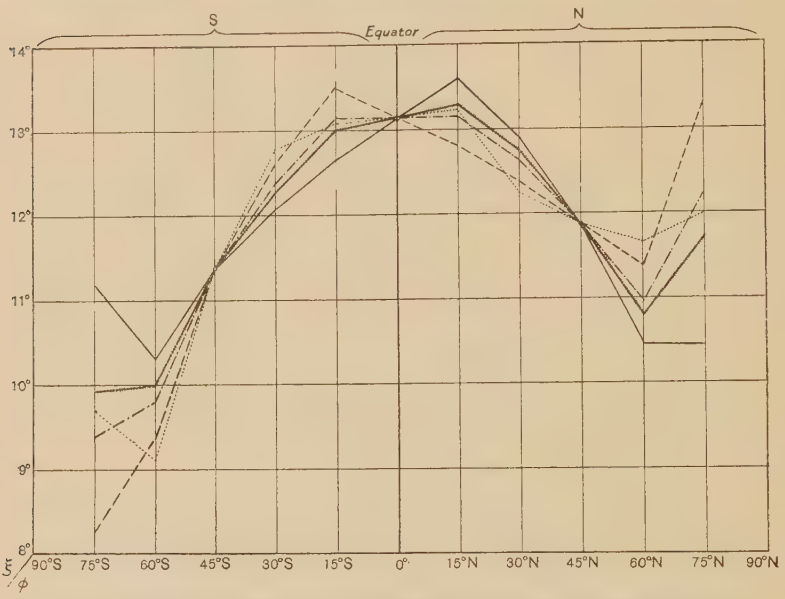


FIG. 4.—Angular velocities, assuming symmetry only with respect to the axis, and taking account of difference between hemispheres as evident from Table V. The velocities for the epoch of observation must lie on a graph approximating to one of the five given.

we recall that the graph in fig. 3 represents the mean of the Northern and Southern parts of each one of the complete graphs of fig. 4.

For selection between the latter we can only make use of *a priori* considerations. But even these do not point convincingly to one of the alternatives as the most probable. It seems as plausible to suppose that the three angular velocities at 15° S., 0°, and 15° N. were equal, so that a broad equatorial belt was at the time of observation rotating at a uniform rate, as to assume a maximum on one side of the equator or the other. It may further seem necessary to eliminate those curves which contain portions where the gradient is too steep, because in observations involving such points the deviations from the mean value would

certainly have come out much larger, when comparing the four series, than in the other observations. Table VI. shows that there is no case of an outstanding range of values for one latitude compared with the mean range for all, if we except the one low value for v_0 in Series IV., where there can be no question of maximum gradient. We have, however, no right to reject curves with steep portions, as these portions may, after all, stand for parts of the curve where there had been sudden jumps and then again normal gradients, in the sense required by the solar theory of Emden.

It is this theory which perhaps affords the best clue towards an interpretation of most of these results. It will be remembered that Emden finds * that in a rotating gaseous sphere under what he calls polytropic equilibrium surfaces of discontinuity must occur which are surfaces of revolution having the solar axis for their axis and which intersect a meridian plane by lines as shown in fig. 5. A layer of the solar mass between two such surfaces has a constant moment of rotation, *i.e.* the angular velocity within

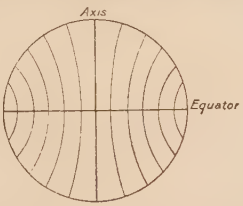


Fig. 5.—Section through sun to show where planes of discontinuity of Emden's theory intersect a meridian plane.

such a layer varies inversely as the square of the radius. Passing through such a surface of discontinuity, outwards from the axis, we pass from a layer with smaller to one with larger moment of rotation. The result is that when plotting the angular velocities from 90°S. to 90° N. we should get, according to this theory, a curve approximating to that given in fig. 6. At the points where surfaces of discontinuity intersect the limb there would be a sudden drop in the angular velocity, while between these points it would *increase* with the latitude in the ratio required by the theory.

We see from the curves between which we must choose in order to conform to our observations that the true one may very well be of this character. All we need assume is that, at the time, the planes of discontinuity were on the whole few and the jumps fairly large. The increase in angular velocity which we would have to find on getting close to the poles is a special feature of our results. Equality of angular velocities at 15°S., 0°, and 15°N., a possible solution of our equations, is also demanded by the theory so long as the surfaces of discontinuity do not crowd together too closely near the equator. A considerable difference between velocities at certain Northern latitudes compared with those at

* R. Emden, *Gaskugeln*, p. 429 *seqq.*

the corresponding Southern latitudes would necessarily be found if, on account of only a slight asymmetry between the hemispheres, in one case the point in question were situated on the equatorial side of a surface of discontinuity, in the other case on the polar side of the same surface. Finally, the symmetry which we have found to exist with respect to the axis is in agreement with the theory, in so far as the surfaces of discontinuity are primarily surfaces of revolution, so that the features which they call into existence would be identical on either side of the limb. The arrangement of velocities in zones along parallels, giving an otherwise irregular retardation towards the poles, recalls the conditions

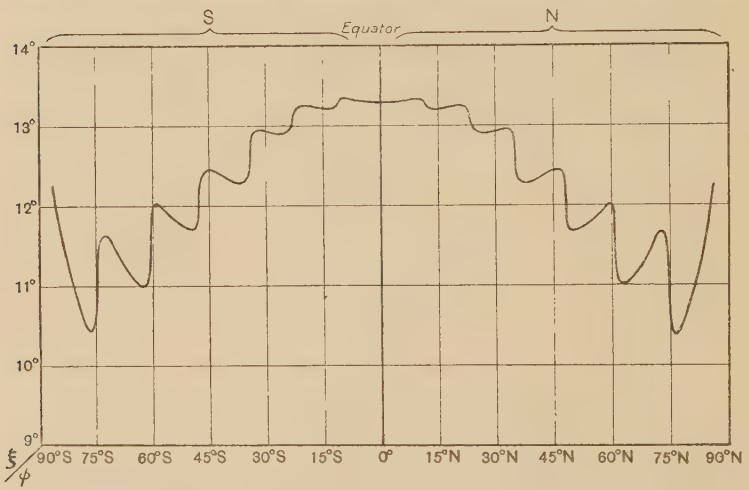


FIG. 6.—Distribution of angular velocities following from Emden's theory of the sun.

found on Jupiter, a planet which resembles the Sun in more than one feature.

Whatever may be said with regard to the above tentative interpretation of the results, the fact of strong deviation from the generally assumed rotation law need not cause surprise. In a careful scrutiny of all earlier investigations on the subject it is possible to find several cases in which at certain definite epochs well-established increases of angular velocity with increase of latitude existed. It is, on the other hand, well known that the various observers have obtained different absolute values for the rotational velocity, even when, as was always done in these investigations, observations were averaged over considerable periods.

The present series of observations may claim definitely to establish that large deviations from the average law occur, which deviations remain constant for periods like that within which these observations were made, and which affect the velocities on solar parallels as a whole.

Summary.

Four series of observations were made during the first fortnight of June 1911, at Cambridge, of which each contains the material, consisting of 48 plates, from which velocity-differences have been derived between pairs of points separated by 90° at intervals of 15° all round the Sun's limb.

The comparison of results derived from different lines in the spectrum examined has brought out the result that the velocity-differences observed, and probably the velocities themselves, vary regularly from one end of the spectral region to the other, becoming smaller on proceeding towards the red. The presumption is that the cause of this anomaly must be looked for in the Sun.

Apart from this effect no special departure from the average has been found characteristic of any one line or any one element.

The results deduced from discussion of the mean velocity-differences per observation may, as far as they are relative, be assumed to be unaffected by the above, and can be summarised under three heads:—

1. The velocities appear to be arranged with great symmetry with respect to the solar axis.

2. With respect to the equator symmetry is conspicuously absent. The sum of the velocities in the Northern hemisphere is considerably greater than that in the Southern, the total difference (for the five latitudes 15°, 30°, up to 75°) amounting to 0·270 km./sec. For latitude 45° the individual difference N.—S. is deduced without ambiguity; it there amounts to 0·052 km./sec.

3. When averaged together to one quadrant the results are shown to be inconsistent with Faye's law or any other simple law. This remains true if, on various hypotheses, the difference between the Northern and Southern hemispheres is allowed for. The conclusion is reached that large deviations from the usual average law occur, which are—to account for what was found under 1—constant all along the respective parallels, and which persist during periods of observation like that of this investigation. The deviations occur especially at very low and at high latitudes. For these zones the angular velocity is found to remain constant or even to increase with increasing latitude.

All these results, inclusive of the difference between the Northern and Southern hemispheres, are consistent with Emden's theory of the Sun. But even if no stress is put upon this interpretation, they merit attention on account of the agreement between the four series of observations. For those given under 3 a considerable amount of confirmation can be found in work by other investigators. Evidence with regard to the conclusions under 1 and 2 cannot be got from other work owing to the hitherto employed method of observation together with the tacit assumptions made in the reductions.

1915 *June* 10.

The Rotation-period of Neptune. By Maxwell Hall, M.A.

Observing Neptune towards the end of November 1883, there were two stars of about the same magnitude in the field of view, and it was noticed that while the light of the stars remained steady through the night, or from night to night, the light of Neptune varied considerably. Comparisons continued for a fortnight or so gave a period of rotation of Neptune of about 7 hr. 55 min., and then the variations suddenly ceased. The observations were published in the *Monthly Notices*, vol. xliv. p. 257.

During the recent opposition of Neptune photometric observations were made; the method adopted is such that if two stars given in the catalogue of the Revised Harvard Photometry are compared, and if the magnitude of one be assumed correct, the average difference of the other is 0·08 (neglecting the sign) from its magnitude as given in the catalogue. The stars compared were of all magnitudes and at all altitudes, and the comparisons were made during 23 nights some years ago. This average error of 0·08 magnitude is due chiefly to variations in the transparency of the atmosphere, to errors of observation, and to what errors there may be in the magnitudes of the stars as given in the catalogue itself.

The recent observations were commenced on February 26, and considerable variation was seen that night; and regular variations were noted up to and including March 30. But on April 7 the regularity broke down; and from May 3 to May 10 inclusive, when the observations ceased, there was no variation at all. From the 12 results obtained during this last interval it will be found that the normal magnitude of Neptune was 7·67, with a mean apparent error of 0·05 (neglecting the signs) and a probable error of ±0·03 for the series, assuming the magnitudes of the comparison stars to be correct.

The comparison stars were:—

	R.A.	Decl.	Magn.
	h m	° ′	
a . . .	7 59·6	+20 3	7·76
b . . .	7 59·3	+20 19	8·37

and these magnitudes were found from μ Cancri, which was taken from the R.H.P. to be 5·38.

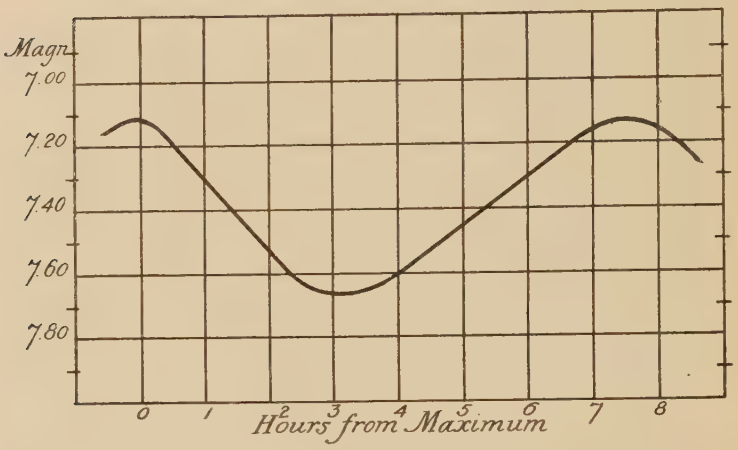
The following table requires but little explanation; local mean time was used, 5 hr. 11½ min. west of Greenwich; and as most of the observations were made at short intervals of time, they were combined together, the second column giving the number of observations combined, and the third column giving the result.

Photometric Observations of Neptune.

1915.	K.M.T.	No. of Obs.	Magn. of N.	Notes.
	d. hr.			
Feb.	26 8·3	3	7·37	Max. 11·0 hr. ; magn. 6·97.
	10·0	3	7·24	
	11·8	3	7·03	
	13·6	4	7·38	
Mar.	2 9·5	3	7·26	Max. 9·0 hr. ; magn. 7·21.
	11·3	3	7·52	
	12·8	3	7·68	
Mar.	8 7·8	2	7·61	N. a little too faint.
Mar.	9 7·8	3	7·76	Max. 5·5 hr.
	9·2	3	7·72	Min. 8·5 hr. ; magn. 7·76.
	12·3	3	7·26	
Mar.	13 10·7	2	6·87	Max. 11·0 hr. : magn. 6·88.
	11·2	2	6·89	
Mar.	19 8·1	2	7·29	Max. 8·5 hr. ; magn. 7·26.
	8·8	2	7·32	
Mar.	30 10·8	3	6·94	Max. 10·8 hr. ; magn. 6·94.
	11·3	3	7·04	
Apr.	4 11·0	2	7·53	N. a little too bright.
Apr.	7 9·2	3	7·39	The max. was due at 6·8 hr. and the min. at 9·8 hr. Regularity broken.
	9·6	3	7·20	
Apr.	29 9·2	3	7·33	The max. was due at 11·6 hr., and N. should have increased in brightness. Regularity remains broken.
	10·0	3	7·53	
	10·7	2	7·47	
May	3 7·2	3	7·62	All variation apparently ceased. Mean magn. May 3 to May 10 inclusive is 7·67, with a mean apparent error of 0·05, and a probable error of ±0·03 for this series.
	7·6	3	7·72	
	8·4	3	7·70	
	9·2	2	7·69	
May	5 7·8	3	7·57	
	8·4	3	7·68	
	9·1	2	7·70	
May	6 9·0	3	7·60	
	10·0	3	7·75	
May	9 8·8	1	7·76	
May	10 8·4	3	7·65	
	9·1	3	7·59	

The following curve shows the periodic changes of magnitude of Neptune; and the actual observations when marked down have to be linked together, when they will be found to follow

Curve of variation.



the curve more or less closely, although they show irregularities in the times of maximum and minimum, as well as in magnitude at these phases.

Epochs of Max.	Intervals Observed.	No. of Periods.	Length of Period.	Intervals Computed.	Difference O − C.
d. hr.	hr.		hr.	hr.	hr.
1915 Feb. 26 11·0	...	...	...	...	...
Mar. 2 9·0	94·0	12	7·8	94·0	0·0
9 5·5	258·5	33	7·83	258·6	−0·1
13 11·0	360·0	46	7·826	360·4	−0·4
19 8·5	501·5	64	7·836	501·4	+0·1
30 10·8	767·8	98	7·835	767·8	0·0

In the last column but one the intervals were computed by taking the period to be 7·835, or 7 hr. 50 min. 6 sec.

For the proper observation of these variations it is highly desirable that they should be more continuous than is possible at one place; and instead of discussing the different points of interest connected with these observations and the resulting curve, it will be better for me to ask the aid of observers in widely differing longitudes during the opposition of Neptune in 1916, and more particularly towards the end of March and the beginning of April, when the planet will be nearly stationary.

Kempshot Observatory, Jamaica:
1915 *May* 17.

The Form of a Rotating Fluid Mass, as disturbed by a Satellite.

By H. Glauert, B.A.

The researches of Darwin, Poincaré, and other writers have dealt very fully with the form assumed by a rotating fluid mass under the influence of a satellite, which describes a circular orbit in the equatorial plane of the primary body. These investigations, however, deal only with the case when there is no relative movement of the fluid, and it is also generally assumed that the two bodies are of equal density. The former assumption is justified by the fact that any slight viscosity of the fluid will eventually result in the subsidence of any relative movement that may exist at an earlier stage of the evolution. The present paper is an investigation of the ellipsoidal form of equilibrium assumed by a frictionless fluid when the two restrictions are withdrawn.

§ 1. *The Hydrodynamical Equations.*

Consider a fluid body of mass M and density ρ, rotating with angular velocity Ω in the presence of a satellite of mass M′, which describes a circular orbit with angular velocity ω in the equatorial plane of the primary body. The form of equilibrium to be sought is an ellipsoid with one axis constantly directed towards the satellite.

Bring the centre of gravity of the primary body to rest by the application of suitable forces and take a set of axes (Mxyz) of which Mz is the axis of rotation and Mx points constantly in the direction of the satellite.

The force function at the point (xyz) of the primary body is

$$F = \Lambda + \frac{M'}{\rho} - \frac{M'}{l^2}x \quad . \quad . \quad . \quad . \quad (1)$$

where Λ is the force function of the primary body at an internal point, l is the distance of the satellite, and

$$\rho^2 = (l - x)^2 + y^2 + z^2.$$

In this formula the approximation has been made of regarding the secondary body as a sphere as regards its effect on the primary. The necessary correction when the satellite assumes an ellipsoidal form will be considered at a later stage (§ 8).

The velocities at the point (xyz) relative to the rotating axes may be taken to be

$$u - K\omega y,\ v + K\omega x,\ w$$

where $K\omega = \Omega - \omega$, and ($uvw$) represent the irrotational disturbance set up by the satellite. As the investigation is concerned only with a state of steady motion, the hydrodynamical equations become

$$\left.\begin{aligned}
\frac{\partial}{\partial x}\left(\mathrm{F}-\frac{p}{\rho}\right) &= (u-\mathrm{K}\omega y)\frac{\partial u}{\partial x}+(v+\mathrm{K}\omega x)\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-v(\mathrm{K}+2)\omega-(\mathrm{K}+1)^2\omega^2 x\\
\frac{\partial}{\partial y}\left(\mathrm{F}-\frac{p}{\rho}\right) &= (u-\mathrm{K}\omega y)\frac{\partial v}{\partial x}+(v+\mathrm{K}\omega x)\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+u(\mathrm{K}+2)\omega-(\mathrm{K}+1)^2\omega^2 y\\
\frac{\partial}{\partial z}\left(\mathrm{F}-\frac{p}{\rho}\right) &= (u-\mathrm{K}\omega y)\frac{\partial w}{\partial x}+(v+\mathrm{K}\omega x)\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}
\end{aligned}\right\} \quad (2)$$

together with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}, \text{ etc.}$$

and the surface condition

$$\left\{(u-\mathrm{K}\omega y)\frac{\partial}{\partial x}+(v+\mathrm{K}\omega x)\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right\}\mathrm{P}=0 \quad . \quad . \quad (3)$$

where $\mathrm{P}=0$ is the equation of the surface.

Integrating the equations of motion

$$\mathrm{F}-\frac{p}{\rho}=\tfrac{1}{2}(u^2+v^2+w^2)+\mathrm{K}\omega(xv-yu)-\tfrac{1}{2}(\mathrm{K}+1)^2\omega^2(x^2+y^2)+2(\mathrm{K}+1)\omega\int(udy-vdx),$$

and so we write

$$u=\frac{\partial\psi}{\partial y} \qquad v=-\frac{\partial\psi}{\partial x}$$

where ψ is a function of x and y only, and $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$. It follows that $w=0$, and we have

$$\frac{p}{\rho}=\mathrm{F}-\tfrac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}+\mathrm{K}\omega\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\psi+\tfrac{1}{2}(\mathrm{K}+1)^2\omega^2(x^2+y^2)-2(\mathrm{K}+1)\omega\psi \quad (4)$$

while the surface condition becomes

$$\frac{\partial\{\mathrm{P},\ \psi-\frac{1}{2}\mathrm{K}\omega(x^2+y^2)\}}{\partial\{x,\ y\}}=0 \quad . \quad . \quad . \quad (5)$$

Writing $\psi=\phi+\frac{1}{2}\mathrm{K}\omega(x^2+y^2)$ we obtain the simpler equations

$$\left.\begin{aligned}
&\frac{p}{\rho}=\mathrm{F}-\tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}-2(\mathrm{K}+1)\omega\phi+\tfrac{1}{2}\omega^2(x^2+y^2)\\
&\frac{\partial(\mathrm{P},\ \phi)}{\partial(x,\ y)}=0\\
&\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+2\mathrm{K}\omega=0
\end{aligned}\right\} \quad (6)$$

To obtain the ellipsoidal solution we write

$$P \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

and derive

$$\phi = -K\omega \frac{b^2x^2 + a^2y^2}{a^2 + b^2}$$

$$u = -K\omega \frac{a^2 - b^2}{a^2 + b^2} y$$

$$v = -K\omega \frac{a^2 - b^2}{a^2 + b^2} x$$

$$\frac{p}{\rho} = F + K^2\omega^2 \frac{2a^2b^2}{(a^2 + b^2)^2}(x^2 + y^2) + 2K\omega^2 \frac{b^2x^2 + a^2y^2}{a^2 + b^2} + \tfrac{1}{2}\omega^2(x^2 + y^2).$$

For equilibrium this expression must be constant over the surface of the ellipsoid.

Now in the force function

$$\Lambda = -\tfrac{3}{4}M(A_0x^2 + B_0y^2 + C_0z^2)$$

where

$$A_0 = \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)^{\frac{3}{2}}(b^2 + \lambda)^{\frac{1}{2}}(c^2 + \lambda)^{\frac{1}{2}}}$$

and

$$\frac{1}{\rho} = \frac{1}{l}\left(1 + \frac{x}{l} + \frac{2x^2 - y^2 - z^2}{2l^2} + \ldots\right).$$

Hence we derive

$$b^2\left\{ -\tfrac{3}{4}MC_0 - \tfrac{1}{2}\frac{M'}{l^3} \right\} = b^2 \left\{ -\tfrac{3}{4}MB_0 - \tfrac{1}{2}\frac{M'}{l^3} + K^2\omega^2 \frac{2a^2b^2}{(a^2 + b^2)^2} + 2K\omega^2 \frac{a^2}{a^2 + b^2} + \tfrac{1}{2}\omega^2 \right\}$$

$$= a^2 \left\{ -\tfrac{3}{4}MA_0 + \frac{M'}{l^3} + K^2\omega^2 \frac{2a^2b^2}{(a^2 + b^2)^2} + 2K\omega^2 \frac{b^2}{a^2 + b^2} + \tfrac{1}{2}\omega^2 \right\} \quad (7)$$

Writing

$$M' = \mu M$$

$$l^3 = Nabc = n^3abc$$

and using the approximate relation

$$\omega^2 l^3 = M(1 + \mu)$$

we derive the fundamental equations

$$\left.\begin{aligned} a^2NS &= \mu(2a^2 + b^2) + (1 + \mu)(a^2 - b^2)\left\{ K^2 \frac{4a^2b^2}{(a^2 + b^2)^2} + 1 \right\} \\ a^2NT &= \mu(2a^2 + c^2) + (1 + \mu)a^2\left\{ K^2 \frac{4a^2b^2}{(a^2 + b^2)^2} + 2K \frac{2b^2}{a^2 + b^2} + 1 \right\} \end{aligned}\right] \quad (8)$$

where

$$a^2S = \tfrac{3}{2}abc(a^2A_0 - b^2B_0)$$

$$a^2T = \tfrac{3}{2}abc(a^2A_0 - c^2C_0).$$

These two equations connect the five variables $\frac{b}{a}$, $\frac{c}{a}$, μ, N, K which define the system. For the evolution of a particular system, μ will be a constant, while the condition of constant angular momentum will give a third relation and so a linear series of ellipsoids.

§ 2. *Solution of the Fundamental Equations.*

When the shape of the ellipsoid is given, the quantities S and T can be accurately determined by means of the elliptic functions E and F. Assuming that $a>b>c$, as will generally be the case, we may write

$$c = a\cos\gamma$$
$$\frac{a^2-b^2}{a^2-c^2} = k^2 = \sin^2\alpha$$
$$b = a\sqrt{1-k^2\sin^2\gamma} = a\Delta,$$

and then we have, according to Darwin (*Scientific Papers*, vol. iii. p. 120),

$$A_0 = \frac{2}{a^3\sin^3\gamma}\cdot\frac{F-E}{k^2}$$
$$B_0 = \frac{2}{a^3\sin^3\gamma}\left\{\frac{E}{k^2(1-k^2)} - \frac{F}{k^2} - \frac{\sin\gamma\cos\gamma}{\Delta(1-k^2)}\right\}$$
$$C_0 = \frac{2}{a^3\sin^3\gamma}\left\{\frac{\Delta\tan\gamma}{1-k^2} - \frac{E}{1-k^2}\right\}$$

so that

$$S = \frac{3\Delta\cos\gamma}{\sin^3\gamma}\left[\frac{F(1+\Delta^2)}{\sin^2\alpha} - \frac{E(\cos^2\alpha+\Delta^2)}{\sin^2\alpha\cos^2\alpha} + \frac{\Delta\sin\gamma\cos\gamma}{\cos^2\alpha}\right]$$
$$T = \frac{3\Delta\cos\gamma}{\sin^3\gamma}\left[\frac{F}{\sin^2\alpha} - \frac{E(2\cos^2\alpha-\Delta^2)}{\sin^2\alpha\cos^2\alpha} - \frac{\Delta\sin\gamma\cos\gamma}{\cos^2\alpha}\right]$$

The fundamental equations are then

$$\left.\begin{aligned} NS &= \mu(2+\Delta^2) + (1+\mu)(1-\Delta^2)\left\{K^2\frac{4\Delta^2}{(1+\Delta^2)^2} + 1\right\} \\ NT &= \mu(2+\cos^2\gamma) + (1+\mu)\left\{K^2\frac{4\Delta^2}{(1+\Delta^2)^2} + 2K\frac{2\Delta^2}{1+\Delta^2} + 1\right\} \end{aligned}\right] \quad (9)$$

For a particular ellipsoid, defined by the values of α and γ, the equations reduce to

$$N = p_1 + p_2\mu + p_3(1+\mu)K^2$$
$$q_1K^2 + 2q_2K + q_3 + mq_4 = 0$$

where $m = \frac{\mu}{1+\mu}$. These equations must give real values for K and positive values for μ and N. When $a > b > c$ the three coefficients p_1, p_2, p_3 are all positive and so the only condition restricting the possible values of μ is

$$mq_1q_4 \leqslant q_2^2 - q_1q_3.$$

When this inequality is satisfied there are two possible values of K, and each gives a corresponding value of N, the larger value of N going with the larger value of K^2.

If, however, it is desired to obtain the form of the ellipsoid corresponding to given values of μ, N, and K, a method of interpolation or continued approximation must be employed.

§ 3. *Method of Approximation.*

Write

$$b^2 = a^2(1-s)$$
$$c^2 = a^2(1-t)$$

where s and t are in general small positive fractions. S and T can then be expressed as series in ascending powers of s and t.

$$a(a^2A_0 - b^2B_0) = a^3\{sB_0 - (B_0 - A_0)\}$$
$$= s\int_1^\infty \frac{d\lambda}{\{\lambda(\lambda-s)^3(\lambda-t)\}^{\frac{1}{2}}} - s\int_1^\infty \frac{d\lambda}{\{\lambda^3(\lambda-s)^3(\lambda-t)\}^{\frac{1}{2}}}$$

giving

$$a(a^2A_0 - b^2B_0) = \tfrac{4}{15}s + \tfrac{4}{35}s(\tfrac{3}{2}s + \tfrac{1}{2}t) + \tfrac{4}{63}s(\tfrac{15}{8}s^2 + \tfrac{3}{4}st + \tfrac{3}{8}t^2)$$
$$+ \tfrac{4}{99}s(\tfrac{35}{16}s^3 + \tfrac{15}{16}s^2t + \tfrac{9}{16}st^2 + \tfrac{5}{16}t^3) + \ldots.$$

Also

$$bc = a^2(1-s)^{\frac{1}{2}}(1-t)^{\frac{1}{2}}$$
$$= a^2\{1 - \tfrac{1}{2}(s+t) - \tfrac{1}{8}(s^2 - 2st + t^2) - \tfrac{1}{16}(s^3 - s^2t - st^2 + t^3) \ldots\},$$

and hence

$$\left.\begin{aligned} S = {} & \tfrac{2}{5}s + \tfrac{2}{35}s^2 - \tfrac{4}{35}st - \tfrac{2}{35}st^2 \\ & - (0{\cdot}01385)s^4 + (0{\cdot}01039)s^3t - (0{\cdot}00519)s^2t^2 - (0{\cdot}03463)st^3 + \ldots. \end{aligned}\right\} \quad (10)$$

while the value of T can be obtained by interchanging s and t in S.

To the third powers of s and t

$$\left.\begin{aligned} S &= \tfrac{2}{35}s(7 + s - 2t - t^2) \\ T &= \tfrac{2}{35}t(7 - 2s + t - s^2) \end{aligned}\right\} \quad \ldots \quad (11)$$

and the fundamental equations (8) can now be written

$$\left.\begin{aligned} \mathrm{NS} &= \mu(3-s)+(1+\mu)s\left\{\mathrm{K}^2\frac{1-s}{(1-\frac{1}{2}s)^2}+1\right\} \\ \mathrm{NT} &= \mu(3-t)+(1+\mu)\left\{\mathrm{K}^2\frac{1-s}{(1-\frac{1}{2}s)^2}+2\mathrm{K}\frac{1-s}{1-\frac{1}{2}s}+1\right\} \end{aligned}\right] \quad (12)$$

From these equations we can obtain the conditions limiting the values of μ and N. Eliminating μ from equations (12) we obtain a quadratic for K, involving N linearly.

$$\mathrm{K}^2\frac{1-s}{(1-\frac{1}{2}s)^2}\{(3-4s+st)+\mathrm{N}(\mathrm{S}-s\mathrm{T})\}+2\mathrm{K}\frac{1-s}{1-\frac{1}{2}s}(3-s+\mathrm{NS}) + \{(3-4s+st)+\mathrm{N}(4\mathrm{S}-3\mathrm{T}-t\mathrm{S})\}=0$$

and for real values of K

$$\mathrm{AN}^2+\mathrm{BN}+\mathrm{C}\geqslant 0$$

where

$$\mathrm{A}=\mathrm{ST}(3+4s-st)-\mathrm{S}^2(3+s-t)-\mathrm{T}^2 3s$$
$$\mathrm{B}=\mathrm{T}(9-9s-4s^2+3st+s^2t)-\mathrm{S}(9-12s-3t-2s^2+9st-st^2)$$
$$\mathrm{C}=(9s-9s^2-6st-s^3+8s^2t-s^2t^2).$$

Retaining only the chief terms of the coefficients, the inequality becomes

$$\tfrac{4}{25}\frac{t-s}{s}(s\mathrm{N}+\tfrac{15}{4})^2+3\left(s-\tfrac{3}{4}\frac{t-s}{s}\right)\geqslant 0 \quad . \quad . \quad (13)$$

and the two values of K coincide when this expression vanishes. This will happen for a positive value of N if

(1) $t<s$

or

(2) $t>s+\frac{4}{3}s^2,\ s<0.$

Again, eliminating N from equations (12) to obtain the limiting conditions for μ, we have

$$\mathrm{K}^2\frac{1-s}{(1-\frac{1}{2}s)^2}(\mathrm{S}-s\mathrm{T})+2\mathrm{K}\frac{1-s}{1-\frac{1}{2}s}\mathrm{S}+(\mathrm{S}-s\mathrm{T})+m\{3(\mathrm{S}-\mathrm{T})+(s\mathrm{T}-t\mathrm{S})\}=0,$$

and for real values of K

$$(1-s)\mathrm{S}^2-(\mathrm{S}-s\mathrm{T})^2\geqslant m(\mathrm{S}-s\mathrm{T})\{3(\mathrm{S}-\mathrm{T})+(s\mathrm{T}-t\mathrm{S})\}.$$

Retaining again only the first powers of s and t, the inequality becomes

$$s^2(2t-s)\geqslant 3ms(s-t) \quad . \quad . \quad . \quad (14)$$

which gives a limit to m in certain cases and serves to indicate the different types of solution and the conditions under which they become possible. The results are given in Table I.

TABLE I.

Form of Ellipsoid.		Necessary Conditions.
$a>b>c$	$1>t>s>0$	No restrictions
$b=c$	$t=s$	$s>0$, *i.e.* $a>b, c$
$a=b$	$s=0$	Always impossible
$a=c$	$t=0$	$s<0$, *i.e.* $b>a, c$. $m<-\frac{1}{3}s$
$a>c>b$	$1>s>t>0$	$2t-s>0$, *i.e.* $a^2+b^2>2c^2$. $m<\frac{s(2t-s)}{3(s-t)}$
$b>a>c$	$1>t>0>s$	$m<-\frac{s(2t-s)}{3(t-s)}$
$b>c>a$	$0>t>s$	$2t-s>0$, *i.e.* $a^2+b^2>2c^2$. $m<\frac{s(2t-s)}{3(s-t)}$
$c>a, b$	$0>t, s>t$	$s<0$, *i.e.* $c>b>a$. $m>-\frac{s(s-2t)}{3(s-t)}$

This table gives the necessary conditions for real values of K, but it is also necessary that the value of N given by the first equation (12) shall be positive. Difficulty on this account can only arise when s is negative, and the additional condition then is

$$(3-s)m<(-s)(K^2+1).$$

This inequality may change the limit for m as given in the table, but the case of $c>b>a$ may become impossible, for there are no real values of m if $K^2(s-t)<-t$. A more accurate general discussion becomes very complicated, but this case can easily be investigated when particular numerical values are given to s and t.

§ 4. *Solution in Series.*

For sufficiently large values of N the quantities s and t may be expressed as series in descending powers of N. As a first step the results for the case of $K=0$ will be obtained, starting from the equations

$$NS=3\mu+s$$
$$NT=1+4\mu-\mu t.$$

Assume

$$s=\frac{a_1}{N}+\frac{a_2}{N^2}+\frac{a_3}{N^3}+\ldots.$$
$$t=\frac{b_1}{N}+\frac{b_2}{N^2}+\frac{b_3}{N^3}+\ldots.$$

and on substituting to order $\frac{1}{N^2}$ in the equations, we derive

$$\tfrac{3}{2}\tfrac{5}{}\left(3\mu+\frac{a_1}{N}+\frac{a_2}{N^2}\right)=7a_1+\frac{1}{N}(7a_2+a_1^2-2a_1b_1)$$
$$+\frac{1}{N^2}(7a_3+2a_1a_2-2a_1b_2-2a_2b_1-a_1b_1^2)$$
$$\tfrac{3}{2}\tfrac{5}{}\left(1+4\mu-\frac{\mu b_1}{N}-\frac{\mu b_2}{N^2}\right)=7b_1+\frac{1}{N}(7b_2-2a_1b_1+b_1^2)$$
$$+\frac{1}{N^2}(7b_3+2b_1b_2-2a_1b_2-2a_2b_1-a_1^2b_1),$$

from which we obtain

$a_1=\tfrac{1}{2}\tfrac{5}{}\mu$ $\quad$ $b_1=\tfrac{5}{2}(1+4\mu)$
$a_2=\tfrac{7}{2}\tfrac{5}{8}\mu(9+5\mu)$ $\quad$ $b_2=-\tfrac{2}{2}\tfrac{5}{8}(1+9\mu+20\mu^2)$
$a_3=\tfrac{3}{3}\tfrac{7}{9}\tfrac{5}{2}\mu(86+101\mu+82\mu^2)$ $\quad$ $b_3=\tfrac{1}{3}\tfrac{2}{9}\tfrac{5}{2}(2+81\mu+430\mu^2+552\mu^3)$.

Thus

$$\left.\begin{aligned} s&=\tfrac{1}{2}\tfrac{5}{}\mu\frac{1}{N}+\tfrac{7}{2}\tfrac{5}{8}\mu(9+5\mu)\frac{1}{N^2}+\tfrac{3}{3}\tfrac{7}{9}\tfrac{5}{2}\mu(86+101\mu+82\mu^2)\frac{1}{N^3}+\ldots.\\ t&=\tfrac{5}{2}(1+4\mu)\frac{1}{N}-\tfrac{2}{2}\tfrac{5}{8}(1+9\mu+20\mu^2)\frac{1}{N^2}+\tfrac{1}{3}\tfrac{2}{9}\tfrac{5}{2}(2+81\mu+430\mu^2+552\mu^3)\frac{1}{N^3}+\ldots.\end{aligned}\right\}\ (1$$

The extension to the case when K has a finite value may now be made. The equations are

$$N(7s+s^2-2st-st^2)=\tfrac{3}{2}\tfrac{5}{}\{3\mu+s+(1+\mu)sK^2\}$$
$$N(7t-2st+t^2-s^2t)=\tfrac{3}{2}\tfrac{5}{}\{1+4\mu-\mu t+(1+\mu)(1-\tfrac{1}{4}s^2)K^2+(1+\mu)(1-\tfrac{1}{2}s-\tfrac{1}{4}s^2)2K$$

Put
$$s=s_0+\Delta s$$
$$t=t_0+\Delta t$$

where s_0, t_0 are the values corresponding to $K=0$. It will appear that Δt is of order $\frac{1}{N}$, Δs of order $\frac{1}{N^2}$, and so the appropriate assumptions are

$$\Delta s=\ldots\frac{a_2}{N^2}+\frac{a_3}{N^3}+\ldots.$$
$$\Delta t=\frac{\beta_1}{N}+\frac{\beta_2}{N^2}+\frac{\beta_3}{N^3}+\ldots.$$

Proceeding as before the results obtained are:—

$\beta_1=\tfrac{5}{2}(1+\mu)K(K+2)$
$a_2=\tfrac{7}{2}\tfrac{5}{8}\mu(1+\mu)K(9K+4)$
$\beta_2=-\tfrac{2}{2}\tfrac{5}{8}(1+\mu)K\{(1+\mu)(K^3+4K^2+6K+4)+7\mu(K+5)\}$
$a_3=\tfrac{3}{3}\tfrac{7}{9}\tfrac{5}{2}\mu(1+\mu)K\{(1+\mu)(86K^3+92K^2+208K+92)$
$\qquad -\mu(71K+30)\}$
$\beta_3=\tfrac{1}{1}\tfrac{2}{5}\tfrac{5}{6}\tfrac{}{8}(1+\mu)K\{8(1+\mu)^2(K^5+6K^4+15K^3+20K^2+15K+6)$
$\qquad +12\mu(1+\mu)(25K^3+23K^2+94K+23)+\mu^2(655K+1226)\}$

so that to order $\frac{1}{N^2}$ the final results become

$$\left.\begin{aligned} s &= \tfrac{15}{2}\mu\frac{1}{N} + \tfrac{75}{28}\mu[(9+5\mu)+(1+\mu)K(9K+4)]\frac{1}{N^2} + \ldots \\ t &= \tfrac{5}{2}[3\mu+(1+\mu)(K+1)^2]\frac{1}{N} \\ &\quad - \tfrac{25}{28}[(7\mu+19\mu^2)+(1+\mu)^2(K+1)^4+7\mu(1+\mu)K(K+5)]\frac{1}{N^2} + \ldots \end{aligned}\right\} \quad (16)$$

It is important to know with what accuracy these series can be used to determine s and t in place of the original equations, and for this purpose a few calculations have been made by the series and also by continued approximation in the equations. The most convenient and accurate method of calculation appears to be the use of the first two terms of the series to obtain values which can then be used as the basis for a closer approximation in the equations. The results of the trial computations are given in Table II.

TABLE II.

			Series		Equations	
μ.	N.	K.	s.	t.	s.	t.
0·5	50	0·0	·0818	·1468	·0817	·1468
	100	0·0	·0391	·0741	·0391	·0741
		0·5	·0401	·1190	·0401	·1192
		1·0	·0422	·1811	·0422	·1813
1·0	50	0·0	·1671	·2420	·1674	·2421
	100	0·0	·0790	·1227	·0790	·1225
		0·5	·0817	·1806	·0817	·1805
		1·0	·0876	·2610	·0879	·2609
2·0	50	0·0	·3502	·4307	·352	·430
	100	0·0	·1614	·2181	·1619	·2179
		0·5	·1698	·2999	·1699	·2997
		1·0	·1894	·4142	·1911	·4133

§ 5. *Form of the Secondary Body.*

Hitherto the form of the primary body only has been considered, but the corresponding equations for the satellite can be at once deduced by writing $\frac{1}{\mu}$ for μ in the various equations and formulæ. Let dashed letters refer to the secondary body. It is then to be noted that

$$N'a'b'c' = l^3 = Nabc,$$

and so if χ be the ratio of the mean radii of the primary and secondary bodies

$$N' = \chi^3 N$$

while

$$\frac{\rho'}{\rho} = \mu\chi^3 \quad . \quad . \quad . \quad . \quad (17)$$

The fundamental equations for the satellite now become

$$\left.\begin{aligned} \mu\chi^3 \mathrm{N}S' &= (3 - s') + (1 + \mu)s' \left\{ \mathrm{K}'^2 \frac{1 - s'}{(1 - \frac{1}{2}s')^2} + 1 \right\} \\ \mu\chi^3 \mathrm{N}T' &= (3 - t') + (1 + \mu) \left\{ \mathrm{K}'^2 \frac{1 - s'}{(1 - \frac{1}{2}s')^2} + 2\mathrm{K}' \frac{1 - s'}{(1 - \frac{1}{2}s')} + 1 \right\} \end{aligned}\right\} \quad (18)$$

and the series for s' and t' are

$$\left.\begin{aligned} s' &= \tfrac{15}{2} \frac{1}{(\mu\chi^3 \mathrm{N})} + \tfrac{75}{28} \{(5 + 9\mu) + (1 + \mu)\mathrm{K}'(9\mathrm{K}' + 4)\} \frac{1}{(\mu\chi^3 \mathrm{N})^2} + \ldots . \\ t' &= \tfrac{5}{2} \{3 + (1 + \mu)(\mathrm{K}' + 1)^2\} \frac{1}{(\mu\chi^3 \mathrm{N})} + \tfrac{25}{28} \{(19 + 7\mu) + (1 + \mu)(\mathrm{K}' + 1)^4 \\ &\qquad + 7(1 + \mu)\mathrm{K}'(\mathrm{K}' + 5)\} \frac{1}{(\mu\chi^3 \mathrm{N})^2} + \ldots . \end{aligned}\right\} \quad (19)$$

A few solutions for the case of $\mathrm{K} = \mathrm{K}' = 0$ are given in Table III.

TABLE III.

N.	μ.	s.	t.	χ^3.	s'.	t'.
100	0·8	·063	·103	1·0	·099	·147
				1·5	·065	·099
				2·0	·048	·074
				4·0	·024	·037
100	1·0	·079	·123	1·0	·079	·123
				1·5	·052	·082
				2·0	·038	·062
				4·0	·019	·031
100	1·2	·095	·142	1·0	·066	·106
				1·5	·043	·071
				2·0	·032	·054
				4·0	·016	·027
200	0·8	·031	·052	1·0	·048	·074
				1·5	·032	·050
				2·0	·024	·037
				4·0	·012	·019
200	1·0	·038	·062	1·0	·038	·062
				1·5	·025	·041
				2·0	·019	·031
				4·0	·009	·015
200	1·2	·046	·072	1·0	·032	·054
				1·5	·021	·036
				2·0	·016	·027
				4·0	·008	·013

§ 6. *Similar Ellipsoids.*

In the general case we have now four equations connecting nine variables, but in particular cases of special interest the number of independent variables can be reduced. In the extensive and detailed work of Darwin and other writers the assumptions $K = K' = 0$ and $\mu\chi^3 = 1$ are made, so that for any particular value of μ there is a linear series of ellipsoids, corresponding to different values of N.

Another case of some interest arises when the two bodies have the form of similar ellipsoids, the equations then reducing to

$$\left.\begin{aligned} NS &= 3\mu + s + (1+\mu)\frac{s(1-s)}{(1-\frac{1}{2}s)^2}K^2 \\ NT &= \mu(3-t) + (1+\mu)\left\{K^2\frac{1-s}{(1-\frac{1}{2}s)^2} + 2K\frac{1-s}{1-\frac{1}{2}s} + 1\right\} \\ (\mu\chi^3 - 1)NS &= (1-\mu)(3-s) - (1+\mu)\frac{s(1-s)}{(1-\frac{1}{2}s)^2}(K^2 - K'^2) \\ (\mu\chi^3 - 1)N(S - sT) &= (1-\mu)(3 - 4s + st) + 2(1+\mu)\frac{s(1-s)}{1-\frac{1}{2}s}(K - K') \end{aligned}\right\} \quad (20)$$

The last two equations show that if the bodies are of equal density it is necessary that $\mu = 1$ and $K = K'$. Again, if we assume that $K = K'$, it follows that the denser body has the smaller mass and volume, and the two bodies must be prolate spheroids. This latter case will next be examined in detail.

If the primary body has the form of a prolate spheroid, so that $b^2 = c^2 = a^2(1-\sigma^2)$, the accurate expression for S or T is

$$S = \tfrac{2}{2}\frac{1-\sigma^2}{\sigma^2}\left\{(1 - \tfrac{1}{3}\sigma^2)\tfrac{1}{2}\log_e\frac{1+\sigma}{1-\sigma} - 1\right\} \quad . \quad . \quad (21)$$

and the fundamental equations become

$$\left.\begin{aligned} NS &= 3\mu + s + (1+\mu)\frac{s(1-s)}{(1-\frac{1}{2}s)^2}K^2 \\ \left(\frac{1-s}{1-\frac{1}{2}s}K + 1\right)^2 &= s \end{aligned}\right\} \quad . \quad . \quad (22)$$

Taking the larger value of K,

$$K = -\frac{1-\frac{1}{2}\sigma^2}{1+\sigma} = -(1-\sigma)(1 + \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\sigma^4 + \ldots)$$

$$NS = 3\mu + \sigma^2 + (1+\mu)\sigma^2(1 - 2\sigma + 2\sigma^2 - 2\sigma^3 + \ldots)$$

which is an equation for determining σ in terms of μ and N by the method of successive approximation.

For the secondary body, in the same way,

$$\mu\chi^3 NS = 3 + \mu\sigma^2 + (1+\mu)\sigma^2(1 - 2\sigma + 2\sigma^2 - 2\sigma^3 + \ldots),$$

and it is now possible to obtain μ in terms of χ and σ, or in terms of χ and N.

Writing $\chi^3 = V^4$ we have the equation

$$\mu V^4\{3\mu + (2+\mu)\sigma^2 - (2+2\mu)\sigma^3 + \ldots.\} = \{3 + (1+2\mu)\sigma^2 - (2+2\mu)\sigma^3 + \ldots.\}$$

which leads to

$$\mu = \frac{1}{V^2} - \tfrac{1}{3}\frac{V^4-1}{V^4}\sigma^2 + \tfrac{1}{3}\frac{V^4-1}{V^4}\sigma^3 + \frac{(V^4-1)(V^4-4V^2-1)}{18V^6}\sigma^4 + \ldots. \quad (23)$$

and $\frac{\rho'}{\rho} = \mu V^4$. The series will be convergent for small values of σ, more particularly when χ^3 is approximately unity.

Eliminating σ from equation (23) by means of the series in descending powers of N,

$$\mu = \frac{1}{V^2} - \tfrac{5}{2}\frac{V^4-1}{V^6}\frac{1}{N} + \tfrac{5}{2}\left(\tfrac{15}{2}\right)^{\frac{1}{2}}\frac{V^4-1}{V^7}\frac{1}{N^{\frac{3}{2}}} - \tfrac{25}{36}\frac{(V^4-1)(7V^4+48V^2+21)}{V^{10}}\frac{1}{N^2} + \ldots.$$

In this type of solution it has appeared that

$$\frac{\Omega}{\omega} = \sigma - \tfrac{1}{2}\sigma^2 + \ldots.,$$

implying that for each body the "day" is very long compared with the "month." A more probable condition and the one usually adopted is that $\Omega = \Omega' = \omega$, for this is the condition which will be brought about eventually by tidal friction on account of the viscosity of the fluid. Now it has been shown that for $K = K'$ the two bodies can be similar only if they are prolate spheroids. It is, however, of some interest to examine how close an approach to similarity of figure can be obtained under the assumption $K = K' = 0$.

Taking only the first terms of the expansion in series

$$\frac{s-s'}{s'} = \mu^2\chi^3 - 1 \qquad \frac{t-t'}{t'} = \mu\frac{1+4\mu}{4+\mu}\chi^3 - 1$$

and when N is large these expressions can be taken to represent the deviation from similarity of the two ellipsoids. It is desired to find the conditions under which each of these expressions may be less than some small quantity ϵ, and to find the least possible value $\bar{\epsilon}$ of ϵ corresponding to a definite value of μ.

Writing

$$\mu^2\chi^3 = 1 - \theta$$

the conditions are

$$-\epsilon < \theta < \epsilon$$

$$-\epsilon\mu(4+\mu) < (1-\mu^2) - \theta(1+4\mu) < \epsilon\mu(4+\mu).$$

The conditions that these two inequalities are not inconsistent lead to the result

$$\epsilon \geqslant \frac{|(1-\mu^2)|}{1+8\mu+\mu^2},$$

and so by definition

$$\bar{\epsilon} = \frac{|(1-\mu^2)|}{1+8\mu+\mu^2} \quad . \quad . \quad . \quad . \quad (25)$$

while the corresponding value of χ^3 is given by

$$\frac{\rho'}{\rho} = \mu\chi^3 = \frac{2(\mu+4)}{1+8\mu+\mu^2}.$$

Numerical values are given in Table IV. The figures show that there can be no close similarity unless μ is very nearly unity.

Table IV.

μ =	0·5	0·6	0·7	0·8	0·9	1·1	1·2	1·3	1·4	1·5
$\bar{\epsilon}$ =	·143	·104	·072	·045	·021	·019	·037	·053	·068	·082
$\frac{\rho'}{\rho}$ =	1·71	1·49	1·33	1·19	1·09	0·93	0·86	0·81	0·76	0·72
χ^3 =	3·43	2·49	1·89	1·49	1·21	0·84	0·72	0·62	0·54	0·48
χ =	1·51	1·36	1·24	1·14	1·10	0·94	0·90	0·85	0·81	0·78

§ 7. *Angular Momentum and Kinetic Energy.*

Having now considered the forms of the two fluid bodies, the next step will be to obtain expressions for the angular momentum and kinetic energy of the system.

The orbital momentum

$$= \mathrm{M}\left(\frac{\mu l}{1+\mu}\right)^2\omega + \mu\mathrm{M}\left(\frac{l}{1+\mu}\right)^2\omega$$

$$= \frac{\mu \mathrm{M}^{\frac{2}{3}} l^{\frac{1}{2}}}{(1+\mu)^{\frac{1}{2}}}$$

and the rotational momentum of the primary ellipsoid

$$= \iiint \rho\omega \left\{ x^2\left(1+\frac{2b^2}{a^2+b^2}\mathrm{K}\right) + y^2\left(1+\frac{2a^2}{a^2+b^2}\mathrm{K}\right)\right\} dv$$

$$= \tfrac{1}{5}\mathrm{M}\omega\left\{ a^2+b^2+\frac{4a^2b^2}{a^2+b^2}\mathrm{K}\right\}.$$

Thus the total angular momentum of the system is

$$\mu\left(\frac{M^3 l}{1+\mu}\right)^{\frac{1}{2}} + \frac{1}{5}\left\{\frac{M^3(1+\mu)}{l^3}\right\}^{\frac{1}{2}} a^2 \left\{2 - s + 2K\frac{1-s}{1-\frac{1}{2}s}\right\}$$
$$+ \frac{\mu}{5}\left\{\frac{M^3(1+\mu)}{l^3}\right\}^{\frac{1}{2}} a'^2 \left\{2 - s' + 2K'\frac{1-s'}{1-\frac{1}{2}s'}\right\}.$$

But if r is the mean radius of the primary body

$$a^2 = r^2(1-s)^{-\frac{1}{3}}(1-t)^{-\frac{1}{3}}$$

and

$$l = nr$$
$$r = \chi r'.$$

On substituting, the expression for the angular momentum becomes

$$\{M^3(1+\mu)r\}^{\frac{1}{2}}A$$

where

$$A = \frac{\mu}{1+\mu}n^{\frac{1}{2}} + \{\tfrac{2}{5}Q(K+1) - \tfrac{1}{10}s^2K\}n^{-\frac{3}{2}} + \mu\chi^{-2}\{\tfrac{2}{5}Q'(K'+1) - \tfrac{1}{10}s'^2K'\}n^{-\frac{3}{2}}. \quad (26)$$

and

$$Q = 1 - \tfrac{1}{6}s + \tfrac{1}{3}t + \tfrac{1}{18}s^2 - \tfrac{1}{18}st + \tfrac{2}{9}t^2 + \ldots.$$

By the use of the series (16) and (19) the ellipticities can be eliminated from the expressions for A and Q. The latter becomes

$$Q = 1 + \tfrac{5}{12}[3\mu + 2(1+\mu)(K+1)^2]\frac{1}{N} + \tfrac{2}{2}\tfrac{5}{5}\tfrac{5}{2}[108\mu^2 + 12\mu(1+\mu)(K+1)^2$$
$$+ 11(1+\mu)^2(K+1)^4]\frac{1}{N^2} + \ldots. \quad (27)$$

Again, the orbital kinetic energy is

$$\tfrac{1}{2}M\left(\frac{\mu l}{1+\mu}\right)^2\omega^2 + \tfrac{1}{2}\mu M\left(\frac{l}{1+\mu}\right)^2\omega^2,$$

i.e.

$$\tfrac{1}{2}\frac{\mu M^2}{l},$$

and the rotational kinetic energy of the primary ellipsoid

$$= \tfrac{1}{2}\iiint \rho\omega^2 \left\{x^2\left(1 + \frac{2b^2}{a^2+b^2}K\right)^2 + y^2\left(1 + \frac{2a^2}{a^2+b^2}K\right)^2\right\} dv$$
$$= \frac{1}{10}\frac{M^2(1+\mu)}{l^3}\left\{\frac{(a^2-b^2)^2}{a^2+b^2} + \frac{4a^2b^2}{a^2+b^2}(K+1)^2\right\}.$$

For the whole system the kinetic energy may now be written as

$$\frac{M^2(1+\mu)}{2r}E$$

where

$$E = \frac{\mu}{1+\mu}\frac{1}{n} + \{\tfrac{2}{5}Q(K+1)^2 - \tfrac{1}{10}s^2(K^2+2K)\}\frac{1}{n^3} + \frac{\mu}{\chi^2}\{\tfrac{2}{5}Q'(K'+1)^2 - \tfrac{1}{10}s'^2(K'^2+2K')\}\frac{1}{n^3} \quad . \quad (28)$$

We can now find the position in which a system of given mass, density, and angular momentum has its maximum kinetic energy, the shape of the two bodies being continually determined by the four fundamental equations. Taking K, K′, and N as the three independent variables, the conditions are

$$dA = dE = 0,$$

and these become

$$K+1)dQ + (\tfrac{2}{5}Q - \tfrac{1}{10}s^2)dK - \tfrac{1}{5}Ksds\}$$
$$+ \frac{\mu}{\chi^2}\{\tfrac{2}{5}(K'+1)dQ' + (\tfrac{2}{5}Q' - \tfrac{1}{10}s'^2)dK' - \tfrac{1}{5}K's'ds'\}$$
$$= \tfrac{2}{2}\frac{dn}{n}\Big[\{\tfrac{2}{5}Q(K+1) - \tfrac{1}{10}s^2K\} + \frac{\mu}{\chi^2}\{\tfrac{2}{5}Q'(K'+1) - \tfrac{1}{10}s'^2K'\} - \tfrac{1}{3}mn^2\Big]$$
$$K+1)^2dQ + (\tfrac{2}{5}Q - \tfrac{1}{10}s^2)2(K+1)dK - \tfrac{1}{5}K(K+2)sds\}$$
$$+ \frac{\mu}{\chi^2}\{\tfrac{2}{5}(K'+1)^2dQ' + (\tfrac{2}{5}Q' - \tfrac{1}{10}s'^2)2(K'+1)dK' - \tfrac{1}{5}K'(K'+2)s'ds\}$$
$$= 3\frac{dn}{n}\Big[\{\tfrac{2}{5}Q(K+1)^2 - \tfrac{1}{10}s^2(K^2+2K)\} + \frac{\mu}{\chi^2}\{\tfrac{2}{5}Q'(K'+1)^2 - \tfrac{1}{10}s'^2(K'^2+2K')\} + \tfrac{1}{3}mn^2\Big]$$

where

$$\tfrac{2}{5}dQ = dK\Big[\tfrac{2}{3}(1+\mu)(K+1)\frac{1}{N} + \tfrac{1}{6}\tfrac{0}{3}(1+\mu)(K+1)\{6\mu + 11(1+\mu)(K+1)^2\}\frac{1}{N^2} + \ldots\Big]$$
$$- \frac{dN}{N}\Big[\tfrac{1}{6}\{3\mu + 2(1+\mu)(K+1)^2\}\frac{1}{N} + \tfrac{5}{6^5_3}\{108\mu^2 + 12\mu(1+\mu)(K+1)^2 + 11(1+\mu)^2(K+1)^4\}\frac{1}{N^2} + \ldots\Big]$$

and

$$sds = -\tfrac{225}{4}\mu^2\frac{1}{N^2}\frac{dN}{N}.$$

With these substitutions the two equations can be written in the form

$$a_1dK + b_1dK' + c_1\frac{dN}{N} = 0$$
$$a_2dK + b_2dK' + c_2\frac{dN}{N} = 0$$

where

$$a_1 = \tfrac{2}{5} + \tfrac{1}{2}\{\mu + 2(1+\mu)(K+1)^2\}\frac{1}{N}$$
$$+ \tfrac{5}{504}\{-135\mu^2 + 144\mu(1+\mu)(K+1)^2 + 220(1+\mu)^2(K+1)^4\}\frac{1}{N^2} + \ldots$$

$$b_1 = \tfrac{2}{5}\frac{\mu}{\chi^2} + \tfrac{1}{2}\{1 + 2(1+\mu)(K'+1)^2\}\frac{1}{\chi^5 N}$$
$$+ \tfrac{5}{504}\{-135 + 144(1+\mu)(K'+1)^2 + 220(1+\mu)^2(K'+1)^4\}\frac{1}{\mu\chi^8 N^2} + \ldots$$

$$c_1 = \tfrac{1}{6}mn^2 - \tfrac{1}{5}\left\{(K+1) + \frac{\mu}{\chi^2}(K'+1)\right\} - \tfrac{1}{4}(K+1)\{3\mu + 2(1+\mu)(K+1)^2\}\frac{1}{N}$$
$$- \tfrac{1}{4}(K'+1)\{3 + 2(1+\mu)(K'+1)^2\}\frac{1}{\chi^5 N} + \ldots.$$

$$a_2 = \tfrac{4}{5}(K+1) + \tfrac{1}{3}(K+1)\{3\mu + 4(1+\mu)(K+1)^2\}\frac{1}{N}$$
$$+ \tfrac{5}{84}(K+1)\{-45\mu^2 + 32\mu(1+\mu)(K+1)^2 + 44(1+\mu)^2(K+1)^4\}\frac{1}{N^2} + \ldots$$

$$b_2 = \tfrac{4}{5}(K'+1)\frac{\mu}{\chi^2} + \tfrac{1}{3}(K'+1)\{3 + 4(1+\mu)(K'+1)^2\}\frac{1}{\chi^5 N}$$
$$+ \tfrac{5}{84}(K'+1)\{-45 + 32(1+\mu)(K'+1)^2 + 44(1+\mu)(K'+1)^4\}\frac{1}{\mu\chi^8 N^2} + \ldots$$

$$c_2 = -\tfrac{1}{3}mn^2 - \tfrac{2}{5}\left\{(K+1)^2 + \frac{\mu}{\chi^2}(K'+1)^2\right\} - \tfrac{1}{3}(K+1)^2\{3\mu + 2(1+\mu)(K+1)^2\}\frac{1}{N}$$
$$- \tfrac{1}{3}(K'+1)^2\{3 + 2(1+\mu)(K'+1)^2\}\frac{1}{\chi^5 N} + \ldots.$$

The position of maximum kinetic energy is now given by the equations

$$a_1 b_2 = a_2 b_1$$
$$a_1 c_2 = a_2 c_1,$$

and to order $\frac{1}{N}$ we have

$$a_1 b_2 = \tfrac{8}{25}\frac{\mu}{\chi^2}(K'+1) + \tfrac{2}{5}(K'+1)\left[\frac{\mu}{\chi^2}\{\mu + 2(1+\mu)(K+1)^2\}\right.$$
$$\left. + \frac{1}{\chi^5}\{1 + \tfrac{4}{3}(1+\mu)(K'+1)^2\}\right]\frac{1}{N}$$

$$a_2 b_1 = \tfrac{8}{25}\frac{\mu}{\chi^2}(K+1) + \tfrac{2}{5}(K+1)\left[\frac{\mu}{\chi^2}\{\mu + \tfrac{4}{3}(1+\mu)(K+1)^2\}\right.$$
$$\left. + \frac{1}{\chi^5}\{1 + 2(1+\mu)(K'+1)^2\}\right]\frac{1}{N}.$$

Putting $K' = K + \frac{k}{N}$ we derive

$$\tfrac{8}{25}\frac{\mu}{\chi^2}k = \tfrac{2}{5}(K+1)\left\{ -\frac{\mu}{\chi^2}\tfrac{2}{3}(1+\mu)(K+1)^2 + \frac{1}{\chi^5}\tfrac{2}{3}(1+\mu)(K+1)^2 \right\}$$

$$K' = K - \tfrac{5}{6}\frac{\mu\chi^3 - 1}{\mu\chi^3}(1+\mu)(K+1)^3\frac{1}{N}.$$

Also

$$a_1c_2 = -\tfrac{2}{15}mn^2 - \tfrac{4}{25}(K+1)^2\left(1+\frac{\mu}{\chi^2}\right) - \frac{m}{6}\{\mu + 2(1+\mu)(K+1)^2\}\frac{1}{n}$$

$$a_2c_1 = \tfrac{2}{15}(K+1)mn^2 - \tfrac{4}{25}(K+1)^2\left(1+\frac{\mu}{\chi^2}\right) + \frac{m}{6}(K+1)\{\mu + \tfrac{4}{3}(1+\mu)(K+1)^2\}\frac{1}{n},$$

giving

$$(\tfrac{4}{5}N + \mu)(K+2) + \tfrac{2}{3}(1+\mu)(K+1)^2(2K+5) = 0,$$

$$K = -2 - \tfrac{5}{6}(1+\mu)\frac{1}{N}$$

$$K' = -2 - \tfrac{5}{6}(1+\mu)\frac{1}{\mu\chi^3 N}.$$

Proceeding to a further approximation by the same method, the results obtained are

$$\left.\begin{aligned} K &= -2 - \tfrac{5}{6}(1+\mu)\frac{1}{n^3} + \tfrac{3}{2}(1+\mu)\frac{\mu\chi^5+1}{\mu\chi^5}\frac{1}{n^5} \\ &\quad - \tfrac{25}{504}(44 + 91\mu + 47\mu^2)\frac{1}{n^6} + \ldots. \\ K' &= -2 - \tfrac{5}{6}(1+\mu)\frac{1}{\mu\chi^3 n^3} + \tfrac{3}{2}(1+\mu)\frac{\mu\chi^5+1}{\mu\chi^5}\frac{1}{n^5} \\ &\quad - \tfrac{25}{504}(47 + 91\mu + 44\mu^2)\frac{1}{\mu^2\chi^6 n^6} + \ldots. \end{aligned}\right\} \quad (29)$$

Substituting these values in the expression for the angular momentum, we obtain an equation to determine the value of n for any given system.

$$\tfrac{3}{2} = -\frac{\mu}{1+\mu}n^2 - \tfrac{2}{5}\left(1+\frac{\mu}{\chi^2}\right) - \tfrac{1}{6}\{(4+7\mu) + \chi^{-5}(7+4\mu)\}\frac{1}{n^3}$$
$$+ \tfrac{3}{5}(1+\mu)\frac{\mu\chi^5+1}{\mu\chi^5}\left(1+\frac{\mu}{\chi^2}\right)\frac{1}{n^5}$$
$$- \tfrac{25}{252}\left\{(128 + 264\mu - 195\mu^2) + \frac{1}{\mu\chi^8}(-195 + 264\mu + 128\mu^2)\right\}\frac{1}{n^6} + \ldots. \quad (30)$$

As an example of the numerical values given by this equation, a series of determinations for the case of $\mu = 0{\cdot}8$, $\chi^2 = 2$ are given.

n =	10	6	5	4	3	2	1·7	1·6	1·5
A =	1·39	1·05	0·94	0·82	0·65	0·34	0·13	0·015	−0·15

In all these cases the value of A derived was unaffected by ignoring the last two terms of equation (30).

§ 8. *Two Corrections to the Fundamental Equations.*

In obtaining the force function at a point of the primary body the satellite has so far been regarded as a sphere and its effect taken to be sufficiently represented by the term $\frac{M'}{\rho}$. A more accurate expression for this part of the force function is

$$F_1 = \frac{M'}{\rho} + \frac{A' + B' + C' - 3I'}{2\rho^3}$$

where

$$A' = \tfrac{1}{5}M'(b'^2 + c'^2), \text{ etc.}$$

$$I' = A'\frac{(l-x)^2}{\rho^2} + B'\frac{y^2}{\rho^2} + C'\frac{z^2}{\rho^2},$$

so that

$$\frac{5I'\rho^2}{M'} = (b'^2 + c'^2)(l-x)^2 + (c'^2 + a'^2)y^2 + (a'^2 + b'^2)z^2$$

$$= a'^2\{2\rho^2 - (s'+t')\rho^2 + (s'y^2 + t'z^2)\}$$

$$I' = \tfrac{1}{5}M'\left\{(2 - s' - t')a'^2 + (s'y^2 + t'z^2)\frac{a'^2}{\rho^2}\right\}.$$

Hence we derive

$$F_1 = \frac{M'}{\rho} + \frac{M'a'^2}{10}\left\{\frac{s'+t'}{\rho^3} - \frac{3(s'y^2 + t'z^2)}{\rho^5}\right\},$$

and expanding the powers of ρ as before, we have for the variable part,

$$\frac{F_1}{\mu M} = \frac{x}{l^2}\left\{1 + \tfrac{3}{10}(s'+t')\frac{a'^2}{l^2}\right\} + \frac{3x^2 - r^2}{2l^3} + \tfrac{3}{20}(s'+t')\frac{a'^2}{l^2}\frac{5x^2 - r^2}{l^3}$$

$$- \tfrac{3}{10}\frac{a'^2}{l^2}\frac{s'y^2 + t'z^2}{l^3} + \ldots,$$

or, to the first powers of s' and t' in the additional terms,

$$\frac{F_1}{\mu M} = \frac{x}{l^2}\left\{1 + \tfrac{3}{10}(s'+t')\frac{1}{\chi^2 n^2}\right\} + \frac{2x^2 - y^2 - z^2}{2l^3}$$

$$+ \tfrac{3}{20}\frac{4(s'+t')x^2 - (3s'+t')y^2 - (s'+3t')z^2}{\chi^2 n^2 l^3},$$

and the last term represents the correction to be applied to the force function. The change in the factor of ν merely alters the force necessary to bring the centre of gravity of the primary body to rest. The corresponding correction to the fundamental equations (12) is the addition on the right-hand sides of the terms

$$\tfrac{3}{10}\frac{\mu}{\chi^2 n^2}(7s' + 5t') \quad \text{and} \quad \tfrac{3}{10}\frac{\mu}{\chi^2 n^2}(5s' + 7t').$$

Using approximate values for s' and t', the corrections to s and t are

$$\left.\begin{aligned} \Delta s &= \tfrac{15}{8}\frac{1}{\chi^5 n^8}\{36 + 5(1+\mu)(K'+1)^2\} \\ \Delta t &= \tfrac{15}{8}\frac{1}{\chi^5 n^8}\{36 + 7(1+\mu)(K'+1)^2\} \end{aligned}\right\} \quad . \quad . \quad (31)$$

The corresponding corrections for the secondary body are

$$\left.\begin{aligned} \Delta s' &= \tfrac{15}{8}\frac{1}{\mu\chi^3 n^8}\{36\mu + 5(1+\mu)(K+1)^2\} \\ \Delta t' &= \tfrac{15}{8}\frac{1}{\mu\chi^3 n^8}\{36\mu + 7(1+\mu)(K+1)^2\} \end{aligned}\right\} \quad . \quad . \quad (32)$$

and for bodies of approximately the same mass and density these corrections are practically negligible.

The second correction to be considered is the modification of the equation $\omega^2 l^3 = M(1+\mu)$ when the ellipsoidal form of the bodies is taken into account. From the form of F_1 obtained above it is apparent the force acting on the centre of gravity of the satellite must be taken to be

$$\frac{\mu M}{l^2}\left(1 + \tfrac{3}{10}\frac{s'+t'}{\chi^2 n^2}\right) + \frac{M}{l^2}\left(1 + \tfrac{3}{10}\frac{s+t}{n^2}\right),$$

so that

$$l^3\omega^2 = M(1+\mu) + \tfrac{3}{10}\frac{M}{n^2}\left\{(s+t) + \frac{\mu}{\chi^2}(s'+t')\right\},$$

which is the equation to replace

$$l^3\omega^2 = M(1+\mu)$$

The necessary modification of equations (12) leads to

$$\begin{aligned} \Delta s &= \tfrac{3}{4}s\frac{K^2+1}{n^5}\left\{(s+t) + \frac{\mu}{\chi^2}(s'+t')\right\} \\ \Delta t &= \tfrac{3}{4}\frac{(K+1)^2}{n^5}\left\{(s+t) + \frac{\mu}{\chi^2}(s'+t')\right\}, \end{aligned}$$

so that it is at once evident that the correction to s is of order n^{-11} and may be neglected. The corrections to t and t' are

$$\left.\begin{aligned}\Delta t &= \tfrac{15}{8}\frac{(K+1)^2}{\chi^5 n^8}\left[6(\mu\chi^5+1)+(1+\mu)\{\chi^5(K+1)^2+(K'+1)^2\}\right]\\ \Delta t' &= \tfrac{15}{8}\frac{(K'+1)^2}{\mu\chi^8 n^8}\left[6(\mu\chi^5+1)+(1+\mu)\{\chi^5(K+1)^2+(K'+1)^2\}\right]\end{aligned}\right\}\quad(33)$$

which are of the same order as those given in equations (31) and (32).

Trinity College, Cambridge:
May 1915.

The Viscosity of the Earth. By Harold Jeffreys, B.A., M.Sc.

The Evidence on the Viscosity of the Solid Earth.

The degree of viscosity within the earth can be estimated in two ways. In the first place, a value of it can be obtained by attributing the empirical part of the secular acceleration of the moon's mean motion to bodily tidal friction. In the second place, more direct experimental evidence can be reached by means of the water pendulum,* which measures directly the lags of the lunar tides. The agreement between the values obtained is remarkably good considering the uncertainty of the results, so that we may be safe in considering the bodily viscosity of the earth to be of the order given by Michelson in spite of his large probable error. The usual effect of viscosity on a free vibration being to cause its amplitude to die down, the problem suggests itself to examine the effect of viscosity on that particular free vibration known as the Eulerian nutation, which is the accepted explanation of the observed phenomenon known as the variation of latitude.

It is actually found that bodily viscosity may have a considerable effect, causing the Eulerian nutation to die down in a very short time. On the other hand, the effect of oceanic viscosity is very small, and the only important influence of the ocean is to lengthen the period somewhat.

I. *The Argument from Tidal Friction.*

If we neglect the tides due to the inclination of the equator and the lunar orbit to the ecliptic, the apparent secular accelera-

* Michelson, *Ap. J.*, vol. xxxix. p. 105.

tion of the moon would be $412''{\cdot}6\mathrm{E}\sin 2\epsilon$ in a century,* due to bodily tidal friction. Here E is the actual amplitude of the semidiurnal *bodily* tide divided by the amplitude of the equilibrium tide in a fluid sphere. ϵ is the angle of lag of the same tide.

If then

v = the angular velocity of the earth relative to the moon,
n = the rigidity of the solid earth,
t = the modulus of relaxation of rigidity of the solid earth, so that
$n\mathrm{t}$ = the coefficient of viscosity used by Darwin and Michelson,
ρ = the density of the earth, supposed uniform,
a = the radius of the earth,
g = the value of gravity at the earth's surface,

$$\frac{1}{k}=\frac{19n}{2g\rho a}+1,$$

$$\tan\chi=2v\mathrm{t}/k\,;\quad\tan\psi=2v\mathrm{t},$$

we have †

$$\mathrm{E}=\cos\chi/\cos\psi$$

and $$\epsilon=\chi-\psi.$$

If t is great, corresponding to strength of considerable permanence, E approximates to k, χ and ψ to $\frac{\pi}{2}$, and ϵ is small.

Further, the ratio of the amplitude of the tide in a liquid layer on a yielding nucleus to the amplitude in a liquid layer on a rigid nucleus is

$$\cos\chi\frac{19vn\mathrm{t}}{g\rho a},$$

while the lag is $\frac{1}{2}\pi-\chi$.

If t is great, the above ratio tends to $1-k$.

This number is taken by Michelson ‡ to be 0·70, making $\mathrm{E}=k=0{\cdot}30$, while $n=8{\cdot}6\times10^{11}$ C.G.S. units.

The most recent value § of the empirical secular acceleration of the moon's mean motion is $4''{\cdot}5$ in a century. The value used by Sir G. H. Darwin was $4''$. The equation

$$412''{\cdot}6\mathrm{E}\sin 2\epsilon=4''{\cdot}5$$

gives then $\epsilon=1^\circ\,5'$. This corresponds to $\frac{1}{2}\pi-\chi=k\epsilon/(1-k)=28'$, so that the apparent acceleration of high water is $0^{\mathrm{h}}{\cdot}031$, inertia being neglected. ||

* Sir G. H. Darwin, *Sci. Papers*, vol. ii. p. 73, equ. (61), or Kelvin and Tait, *Nat. Phil.*, Appendix G.

† *Ibid.*, p. 19.

‡ Michelson, *Ap. J.*, vol. xxxix. p. 126.

§ Fotheringham and Longbottom, *M.N.*, *R.A.S.*, vol. lxxv. p. 396.

|| For the justification of the neglect of inertia, see Darwin, *Sci. Papers*, vol. ii. p. 22.

This value agrees well with Michelson's adopted value, namely, $0^h{\cdot}026$. The closeness is, however, largely accidental, as the effects of heterogeneity have been omitted, while the inclination terms may alter the value of the secular acceleration by $\frac{1}{4}$ of its amount. Further, Michelson's probable error is about $0^h{\cdot}02$.

Michelson's investigation makes $t = 1{\cdot}26 \times 10^5$ seconds. I find that the value of t corresponding to an acceleration of high water of $0^h{\cdot}031$ is $\cdot 87 \times 10^5$ seconds.

II. *The Effect of Bodily Viscosity on the Eulerian Nutation.*

For this purpose the material of the earth is supposed to obey the elastico-viscous law used by Sir G. H. Darwin. We require first a general consideration on the equations of motion in such a medium.

In all motions in incompressible elastic, elastico-viscous, or perfectly viscous media the equations of motion are of the form

$$-\frac{\partial p}{\partial x} + c\nabla^2 u + \rho X = 0, \text{ etc.},$$

where inertia is neglected, and

u, v, w are the component displacements,
X, Y, Z are the component forces per unit mass,
ρ is the density,
p is the mean of the three principal stresses,

and $c = \mu \dfrac{d}{dt}$ in a perfectly viscous medium, μ being the coefficient of viscosity,

$c = n$, the coefficient of rigidity, in a perfectly elastic medium,

$$c = n\left(\frac{d}{dt} + \frac{1}{t}\right)^{-1}\frac{d}{dt} \text{ in an elastico-viscous medium}$$

$$= n\left(1 + \frac{1}{t\dfrac{d}{dt}}\right)^{-1}.$$

In all cases the equation of continuity has the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The stresses across the co-ordinate planes at any point are given in all three cases by

$$P = p + 2c\frac{\partial u}{\partial x}, \text{ etc.}$$

$$S = c\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \text{ etc.}$$

As all the equations of continuity at the boundary in displacement and stress are linear in u, v, w, and p, it follows that if we have solved the problem of the strain produced by a given external potential in a perfectly elastic medium of rigidity n, then the solution in a perfectly viscous or elastico-viscous medium is obtained directly from this by replacing n by $\mu\frac{d}{dt}$ or $n\left(1+\frac{1}{td/dt}\right)^{-1}$ respectively. In particular, if the disturbing potential be a periodic one of the type $A\cos pt$, we can write this $R[Ae^{ipt}]$ where $R(\lambda)$ denotes "the real part of λ," and then the solution is obtained by replacing n by μip or $nti p/(1+ti p)$ and taking the real part of the whole result.

Now Michelson's results give

$$n = 8{\cdot}6\times10^{11} \text{ C.G.S., and } nt = 10{\cdot}9\times10^{16} \text{ C.G.S. units} \quad (1)$$

These are based on the hypothesis that the earth is homogeneous. Let us then consider what will be the period and modulus of decay of the Eulerian nutation if these results are accepted.

The ellipticity caused by centrifugal strain in a homogeneous elastic sphere * of rigidity c is $\frac{\frac{1}{2}\omega^2\cdot 5a/2g}{19c/2a\rho g+1}$, and if the number of days in the period of the Eulerian nutation is τ, while the ellipticity of the Maclaurin ellipsoid corresponding to the actual angular velocity of rotation is $1/\tau_0$, we have

$$\frac{1}{\tau_0} = \frac{5a\omega^2}{4g} = \frac{1}{232}$$

and

$$\frac{1}{\tau} = \frac{1}{\tau_0} - \frac{5a\omega^2}{4g}\Big/\left(1+\frac{19c}{2g\rho a}\right)$$

$$= \frac{5a\omega^2}{4g}\cdot\frac{19c}{2g\rho a}\Big/\left(1+\frac{19c}{2g\rho a}\right).$$

Then

$$\frac{\tau_0}{\tau} = \frac{19c}{2g\rho a}\Big/\left(1+\frac{19c}{2g\rho a}\right) \quad . \quad . \quad . \quad . \quad (2)$$

Now c in this case is to be replaced by $nti p/(1+ti p)$. (3)

and

$$p = 2\pi/86400\tau, \text{ the unit of time being a second} \quad . \quad (4)$$

Also

$$19n/2g\rho a = 2{\cdot}37 \quad . \quad . \quad . \quad . \quad (5)$$

These values are found to lead to

$$p = {\cdot}202i + {\cdot}019 \quad . \quad . \quad . \quad . \quad (6)$$

and therefore the Eulerian nutation will die down according to the law $e^{-{\cdot}202(t-t_0)}\cos{\cdot}019\,(t-t_0)$, where t is measured in days. This

* Kelvin and Tait, *Nat. Phil.*, Art. 840.

means that in 5 days with Michelson's data the amplitude of the variation of latitude should die down to e^{-1} of its original value, while the apparent period (*i.e.* twice the interval between consecutive zeros) would be 329 days.

A homogeneous earth with the velocity of rotation of the actual earth and the ellipticity of the Maclaurin ellipsoid would have a Eulerian period of 232 days, if it was perfectly rigid. Michelson's value of the elasticity, the earth being regarded as perfectly elastic, would also give a period of 329 days.

Thinking that some fallacy might underlie the use of the complex value to replace the operator c in finding the period, I have worked out the whole from first principles, and get the same equation for τ as that used here.

III. *Possible Explanations.*

As the variation of latitude has been observed to exist without any great change in the order of the amplitude for an interval exceeding twenty years, while the amplitude has been observed to remain sensibly constant for intervals of several years, it is obvious that there must be a fallacy in a theory that would require it to die down in a few days. Three suggestions immediately arise as possible explanations.

(1) The law of elastico-viscosity used in Section II. may not be the actual one giving the internal frictional forces. It looks probable, however, that these forces must be either (*a*) independent of the velocities in magnitude, but determined in direction by them; (*b*) proportional to the velocities; or (*c*) dependent on powers of the velocities higher than the first. In both tidal problems (tidal height a few feet) and the variation of latitude the strains in any given portion of the earth's material are so small that it seems incredible that the second and higher powers of the velocities can be considerable. Alternative (*b*) gives practically the same results as the law of elastico-viscosity already chosen. The modulus of decay of an oscillation is presumably determined in its order by the ratio of frictional force to velocity. Now on alternative (*a*) the frictional forces would be of the same magnitude in both the semidiurnal tide and the variation of latitude, while the velocities are proportional to $\frac{3}{2}\frac{m}{c^3}.2v$ and $0''{\cdot}3\omega^2 p$, since the disturbing potentials are $\frac{3}{2}\frac{m}{c^3}r^2(\cos^2\theta-\frac{1}{3})$ and $-\omega^2(lx+my)z$ respectively. This makes the velocities in the semidiurnal tide 250 times as great as in the variation of latitude, so that the determining ratio is vastly greater in the variation of latitude than in the semidiurnal tide. Even if the modulus of decay depends on the ratio frictional force/amplitude the ratio is only reduced to 18, so that alternative (*a*) makes matters worse instead of better.

A further possibility is that on account of the greater size of the tidal strains these may cause the materials of the earth to

surpass their elastic limit, while the strains involved in the variation of latitude would not reach that limit. There are, however, two objections to this. In the first place, the strain in the bodily tide is of the order of 3 feet in 4000 miles, or 1 in 7,000,000. Such a strain as this would not cause permanent set in any known solid of high rigidity, such as the centre of the earth is supposed to be. In the second place, the two systems of stresses are not acting at different times, but are superposed. Consequently, any set that may occur is due to the two jointly, and if the frictional force produced is determined in direction by the resultant of the velocities in the two systems, we may expect it to affect both in proportion to their magnitudes. Thus even in this case there is no great reason to doubt that the total effects of internal friction in the two cases will be adequately described if we regard the frictional forces as linear in the velocities.

The non-linearity of the law of viscosity, then, does not seem to help us much unless we are prepared to accept alternative (*c*), which would require the frictional forces to be proportional to at least the cube of the small velocities involved.

(2) As Michelson's lag is obtained as the mean * between $+0^h{\cdot}059$ and $-0^h{\cdot}007$, and he imposes a considerable probable error on both of these quantities, a very natural course would be to attribute most of his lag to experimental error, were it not strongly confirmed by the empirical secular acceleration of the moon, which is much more accurately known and which has never been explained by anything but tidal friction. The accuracy of the ancient records on which the secular acceleration is based has, however, frequently been questioned, and if they should be regarded as unsatisfactory the estimate of the viscosity will fail with them. Further, these same records give an indication of a secular acceleration of the sun, whose ratio to that of the moon is much greater than one would expect on the theory of tidal friction. This is an additional difficulty in regarding tidal friction as an adequate explanation of the secular acceleration.

The existence of a free precession seems to require, then, not only that the viscosity determined from the secular acceleration shall be incorrect, but that the actual value shall be so high, and the elasticity so perfect, that tidal friction can at present be of hardly any perceptible importance at all. This has no reference, of course, to any influence it may have had in the past, when the whole of the earth may have been much hotter and softer than it is now.

(3) The observed variation of latitude may not be identical with the Eulerian nutation. This suggestion has arguments on both sides. It seems to require the variation of latitude to be a forced vibration of some kind, presumably of seismic character. This may be supported by certain unexplained changes in its amplitude that have taken place since its discovery, without any evident external cause. Further, on the accepted Eulerian theory

* Michelson, *Ap. J.*, vol. xxxix. p. 122.

the empirical secular acceleration seems inexplicable, while this seismic suggestion would leave room for it. On the other hand, the constancy of the period, which bears no very evident relation to any other well-known period in our system, is difficult to account for on any seismic hypothesis.

The continued existence of the Eulerian nutation would seem to show that the modulus of relaxation of rigidity must be many years, which implies that the hot and highly compressed interior of the earth must behave as if nearly perfectly elastic. On the other hand, the estimates of the viscosity from tidal observations and from the moon's secular acceleration give times of relaxation of about a day. If ordinary substances had such short times of relaxation, it is difficult to see how any bridge could continue to stand. These estimates of the viscosity mean, then, that the interior of the earth must be strained nearly to its elastic limit or else that the internal heat makes the material very plastic. These are, then, the physical interpretations of the two discordant results, namely, the existence of the variation of latitude and the empirical secular acceleration.

None of the explanations offered, then, seems wholly satisfactory. Further experimental work may help to clear up the difficulty, especially on the following lines:—

(1) The repetition of Michelson's investigation in more places and with greater accuracy, as he himself suggests.* This may or may not confirm his preliminary estimate of the viscosity, and may perhaps give a new value of the secular acceleration.

(2) The velocities of earthquake waves may be used to determine the rigidity and compressibility of the earth's interior, and from these the theoretical period of the Eulerian nutation, and so to test whether this agrees with the observed period.

IV. *The Effect of Viscosity in the Ocean.*

For this purpose the earth is supposed to consist of a perfectly rigid sphere covered by a shallow ocean. As a preliminary step the ocean may be replaced by a perfectly elastic layer of rigidity c. Then let

u, v, w be the component displacements, and put (1)

$$ux + vy + wz = q. \tag{2}$$

Let the equation of the outer surface undisturbed by any forces be $r = a$,

and after deformation $r = a + q(a)/a$. (3)

At the inner surface $r = \alpha a$. (4)

Then $\alpha = 1 - \beta$, where β is small. (5)

* Michelson, *Ap. J.*, vol. xxxix. p. 127.

Let the potential of the disturbing forces be K, which is equal to $\Sigma k_n K_n(xyz)$ where k_n is a constant and $K_n(xyz)$ is a solid harmonic of degree n. (6)

Also put $q = \Sigma q_n K_n(xyz)$ where q_n is a function of r only. (7)

Then the gravitation potential of the terrestrial spheroid after deformation is

$$V = 4\pi f\left(\frac{1}{r}\int \rho r^2 dr + \int_r^a \rho r dr\right) + 4\pi f \Sigma_n \left\{ \frac{1}{2n+1} \rho q_n(a) K_n(xyz) \right\}$$
$$= \Gamma(r) + \Sigma \kappa_n(r) K_n(xyz) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where f is the constant of gravitation, and ρ = the density at any point.

If then

$$R = \Sigma_n(k_n + \kappa_n) K_n(xyz) \quad . \quad . \quad . \quad (9)$$

the total potential is

$$V + K = \Gamma(r) + R,$$

where R is small of the order of the displacements.

The equations of strain are then found to reduce to

$$\left.\begin{aligned} c\nabla^2 u + \frac{\partial p}{\partial x} + \rho \frac{\partial R}{\partial x} &= 0 \\ c\nabla^2 v + \frac{\partial p}{\partial y} + \rho \frac{\partial R}{\partial y} &= 0 \\ c\nabla^2 w + \frac{\partial p}{\partial z} + \rho \frac{\partial R}{\partial z} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

where the mean pressure $= \int_r^a \rho d\Gamma(r) + p$.

We find that a possible solution is

$$q_n = \frac{A}{r^{2n-1}} + \frac{B}{r^{2n+1}} + Cr^2 + D \quad . \quad . \quad . \quad (11)$$

where A, B, C, D are arbitrary constants.

Then

$$u = \left\{ -\frac{n-2}{n(n+1)} \frac{A}{r^{2n-1}} - \frac{1}{n+1} \frac{B}{r^{2n+1}} + \frac{n+3}{n(n+1)} Cr^2 + \frac{D}{n} \right\} \frac{\partial K_n}{\partial x}$$
$$+ \left\{ \frac{2n-1}{n+1} \frac{A}{r^{2n+1}} - \frac{2C}{n+1} + \frac{2n+1}{n+1} \frac{B}{r^{2n+3}} \right\} x K_n \quad (12)$$

v and w = symmetrical expressions.

Further, if F, G, and H be the component strains across a plane perpendicular to the radius vector,

$$F = \frac{x}{r}\int_r^a \rho d\Gamma + \sigma_n \frac{\partial K_n}{\partial x} + \tau_n x K_n$$

$$G = \frac{y}{r}\int_r^a \rho d\Gamma + \sigma_n \frac{\partial K_n}{\partial y} + \tau_n y K_n$$

$$H = \frac{z}{r}\int_r^a \rho d\Gamma + \sigma_n \frac{\partial K_n}{\partial z} + \tau_n z K_n$$

where

$$\sigma_n = \frac{C}{n(n+1)r}\left\{2(n^2-1)\frac{A}{r^{2n-1}} + 2n(n+2)\frac{B}{r^{2n+1}} + 2n(n+2)Cr^2 + 2(n^2-1)D\right\}$$

$$\tau_n = -\frac{2C}{r}\left\{\frac{(2n-1)(n+2)}{n+1}\frac{A}{r^{2n+1}} + \frac{(2n+1)(n+2)}{n+1}\frac{B}{r^{2n+3}} + \frac{2(n+1)^2+1}{n(n+1)}C\right\} - \frac{\rho}{r}(k_n + \kappa_n)$$

We have in this case

$$\kappa_n = \frac{4\pi f\rho}{2n+1} q_n(a),$$

while if g be the intensity of gravity at the surface,

$$d\Gamma = -g dr.$$

The boundary equations are that at $r = a$,

$$\sigma = 0 \text{ and } a^2\tau_n = -\rho g q_n(a) \quad . \quad . \quad . \quad (14)$$

while at $r = \alpha a$, all the displacements are 0, so that

$$\left.\begin{aligned} -\frac{n-2}{n(n+1)}\frac{A}{r^{2n-1}} - \frac{1}{n+1}\frac{B}{r^{2n+1}} + \frac{n+3}{n(n+1)}Cr^2 + \frac{D}{n} &= 0 \\ \frac{2n-1}{n+1}\frac{A}{r^{2n+1}} - \frac{2C}{n+1} + \frac{2n+1}{n+1}\frac{B}{r^{2n+3}} &= 0 \end{aligned}\right\} \quad (15)$$

The above results are mainly quoted from Herglotz,* who solves the problem for any rigidity and radius of the inner sphere. When $\alpha \to 1$, however, all the four coefficients in Herglotz's final equations (13) become infinite of the second order, and it is easier to solve from first principles than to simplify his equations.

In the case of $n = 2$, then,

$$-\frac{B}{3a^5a^5} + \tfrac{5}{6}Ca^2a^2 + \tfrac{1}{2}D = 0 \quad . \quad . \quad . \quad (16)$$

$$\frac{A}{a^5a^5} - \frac{2C}{3} + \frac{5}{3}\frac{B}{a^7a^7} = 0 \quad . \quad . \quad . \quad (17)$$

$$\frac{6A}{a^3} + \frac{16B}{a^5} + 16Ca^2 + 6D = 0 \quad . \quad . \quad . \quad (18)$$

$$\frac{2c}{a}\left\{\frac{4A}{a^5} + \frac{20}{3}\frac{B}{a^7} + \frac{19}{6}C\right\} + \frac{\rho}{a}\left\{k_n + \tfrac{4}{5}\pi f\rho q_2(a)\right\} = \frac{\rho g}{a^2}q_2(a) \quad (19)$$

* *Zeitsch. f. Math. u. Phys.*, Bd. lii., 1905, p. 278 *et seq.*

Also we have

$$\frac{A}{a^3}+\frac{B}{a^5}+Ca^2+D=q_2(a) \quad . \quad . \quad . \quad (20)$$

For shortness put $q_2(a)=q$.

When α is nearly unity, β is small and the solution of (16), (17), (18), and (20) approximates to

$$A=-\frac{5}{2}\frac{a^3q}{\beta^3};\ B=\frac{15}{14}\frac{a^5q}{\beta^3};\ C=-\frac{15}{14}\frac{q}{a^2\beta^3};\ D=\frac{5}{2}\frac{q}{\beta^3}.$$

If the mean density of the sphere be $\rho(1+\eta)$, $g=\frac{4}{3}\pi f\rho a(1+\eta)$. The remaining equation (19) is then

$$\rho\left(k_2+\frac{3}{5}\frac{g}{a(1+\eta)}q\right)-\frac{\rho g}{a}q=-2c\left(\frac{4A}{a^5}+\frac{20}{3}\frac{B}{a^7}+\frac{19}{6}C\right)$$

$$=+\frac{25}{2\beta^3}\frac{cq}{a^2},$$

or

$$q\left\{\frac{\frac{2}{5}+\eta}{1+\eta}\frac{g}{a}+\frac{c}{\rho a^2}\frac{25}{2\beta^3}\right\}=k_2.$$

Write this in the form $q=\lambda k_2$.

It follows, then, that if as before the actual period of the Eulerian nutation be τ days, while the precessional constant of the whole system is $\frac{1}{\tau_0}$,

$$\frac{1}{\tau}=\frac{1}{\tau_0}-\frac{\frac{1}{2}\omega^2}{\lambda}\frac{1}{1+\eta}.$$

In the last expression, the last term is the value of $(C-A)/C$ induced by the strain of rotation, and the last factor $1/(1+\eta)$ is included to allow for the difference in density between the water and the solid earth. This equation will then give the period of the free precession of a rigid core surrounded by a yielding layer of thickness $a\beta$.

Now, to obtain the effect of the ocean, let us regard it as a uniform layer of viscosity $\mu=\cdot0109$,* and put $\beta=10^{-3}$, so that the depth is about 4 miles. Then to a sufficient approximation we can put

$$\eta=4\cdot5.$$

$$g=981 \text{ cms./(1 sec.)}^2.$$

$$c=\frac{2\pi}{86400\tau}\cdot0109\iota=7\cdot93\times10^{-7}\frac{\iota}{\tau}.$$

$$\frac{a\omega^2}{g}=\frac{1}{289}.$$

$$\tau_0=305.$$

* Lamb, *Hydrodynamics*, p. 536. This is the viscosity of water at 17° C.

Then the equation to be solved is

$$\frac{1}{\tau} = \frac{1}{\tau_0} - \frac{\frac{1}{2}a\omega^2/g}{\frac{2}{5} + \eta + \frac{c}{g\rho a}\frac{25}{2\beta^3}(1+\eta)} \quad . \quad . \quad (21)$$

giving

$$\frac{1}{\tau} = \cdot 00293 + 1\cdot 82 \times 10^{-14}\iota.$$

In the case where the ocean is very shallow, it looks possible at first that the velocities may be considerable and that the neglect of inertia and second order terms is therefore not justifiable. In this case, however, q is of order β^3, and the tangential displacements are of order β^2 and therefore small. The neglect is then justifiable, as the terms in question must be of order β^2/τ^2 and β^4 respectively.

The effect of the ocean is then twofold.

(1) The period is lengthened from 305 days to 342 days. A similar result, allowing for the continents and the flexibility, was obtained by Sir J. Larmor in a paper recently read before the London Mathematical Society.

(2) The free oscillations of the system will die down very slowly, but the imaginary part of $\frac{1}{\tau}$ is so small that no great change in the amplitude can be produced in 10^{10} years.

A considerable change can be made to take place if the ocean is much shallower. If $\beta = 10^{-6}$, making the depth about 20 feet, it is found that the oscillations may die down in about 200 years. In this case, however, the inertia terms may rise into importance. If β is even smaller the whole of the part due to the ocean becomes very small, and $\tau = \tau_0$, the ocean behaving like a rigid body attached to the solid earth.

Next, consider the effect of oceanic viscosity on the semidiurnal tide. We still have the form

$$q = \lambda k_2,$$

but c is now $2 \times 7\cdot 93 \times 10^{-7}\iota$.

Take $\beta = 10^{-3}$.

λ is then found to be $7\cdot 288 \times 10^5(1 - 3\cdot 17 \times 10^{-8}\iota)$, and the lag of the oceanic tide is therefore found to be $4^s\cdot 37 \times 10^{-4}$.

This quantity is so small that the effect of the oceanic tide on the moon must be practically zero, while the effect of the load cannot produce a bodily tide with any appreciable lag, and consequently it is impossible to attribute any measurable part of the observed lag of the bodily tide to oceanic viscosity.

I must express my thanks to Professors Sir Joseph Larmor, Eddington, and Newall for reading through this paper before publication and suggesting numerous improvements.

The Magnitude Scales of some Volumes of the Astrographic Catalogue. By R. J. Pocock, B.A., B.Sc.

The publication of Chapman and Melotte's photographic scale (*Mem. R.A.S.*, vol. lx.) enables us to determine the photographic magnitudes corresponding to measured diameters on the plates of the Astrographic Catalogue by simple counts of the numbers of stars recorded of a given diameter, allowance being made for the personality of the measurers, assuming that each zone is a fair sample of the whole sky.

This has been done in the case of the Perth Catalogue (*M.N.*, *R.A.S.*, lxxv. p. 144) and the Cape Catalogue (*M.N.*, *R.A.S.*, lxxv. p. 57); but in each case the area of the plate was assumed to be 4° instead of the true area $\left(\frac{13}{12} \times 2\right)^2$ or 4°·69, and in neither case was any allowance made for the variations in the star density in different parts of the plate. Owing to this latter cause a given magnitude will be represented by different diameters in different parts of the plate. This effect has been carefully investigated in the case of the Greenwich plates, and the results are given in *M.N.*, *R.A.S.*, lxiii. p. 518. From Tables I., II. of that paper we find the following results—the total number of stars counted in 126 squares of diameter 15 and over is 6372, and of diameter 4 and over is 22,144: taking the means of the three richest squares, the numbers would be, if the density were uniform and equal to the maximum, 6968 and 24,570 respectively. The differences 596, 2426 are in each case equivalent to about 10 per cent., *i.e.* the proportional loss (except possibly for the very bright stars) is approximately independent of the magnitude.

By "loss" it is not intended to signify that the star-images are not there, but that when all diameters not less than 15 (say) are counted, some stars whose magnitudes would entitle them to be included are lost from the counts because their diameters are less than 15. If the diameter d corresponds to magnitude m_1 in the position of best focus, the number counted will be less than the theoretical number given by the C. and M. scale for magnitude m_1. If m_2 corresponds to d in the position of worst focus, we have in the case of the Greenwich results (*loc. cit.*, Table IV.)

$d =$	50	30	20	15	11	8	6	5	4
$m_1 - m_2 =$	1·15	·67	·38	·25	·24	·18	·21	·23	·39

so that for a considerable range $m_1 - m_2$ is practically constant = ·22.

For the brighter stars where the variations are large we need not use the Astrographic Catalogue at all, as there are plenty of other sources available for these stars.

In what follows we will assume that the magnitude represented by d is m_1, viz. its value at the position of best focus. The number N_1' actually counted is less than the number N_1 derived from the C. and M. scale for magnitude m_1, and this fact may be allowed

for by applying a correction to the area of the plate, reducing it to such a value that N_1' will be equal to the theoretical number for such an area. We have seen that the proportional decrease in N_1' and hence in the area is independent of the magnitude. We may therefore determine the area from the total number of stars on the plate. In the case of the Oxford plates this can be done from results given in *M.N.*, *R.A S.*, lxii. p. 434.

The total number per plate is 260 and the maximum number in one square is 84/180, hence uniformly good definition would give a total of $\frac{26^2 \times 84}{180}$, *i.e.* 315. Hence the number counted corresponds to an area of $\frac{260}{315} \times 4^\circ{\cdot}69$, *i.e.* $3^\circ{\cdot}85$. This result may not be strictly applicable to other observatories, but the results will probably be of the same order: if so, the comparison between the C. and M. scale and the Cape and Perth scales as published are very nearly correct, the two points neglected having opposite effects of nearly equal amounts.

If, however, we consider the magnitude represented by d to be that corresponding to the mean of the values which d represents in different parts of the plate, the result will differ somewhat.

In this case there will be about as many stars assessed too bright as are assessed too faint, and we need not apply any correction, but must use the true area covered by the plate, viz. $4^\circ{\cdot}69$.

In the case of the Perth scale magnitude M corresponds to 12·85 on the latter assumption instead of to 13·08 on the assumption of a plate 4° square: for brighter stars the difference is slightly less.

In the case of the Cape scale the values for the faintest stars are 13·45 and 13·19 respectively.

The scales for these two observatories as published in *Monthly Notices* must therefore be considered as giving the values of the letters A to M in the case of Perth and of the diameters in the case of the Cape in the position of best focus. The comparisons which follow have been made on the supposition of a plate area 4° square, so that they are comparable with those already given for Perth and the Cape.

Counts of stars for Toulouse, Potsdam, and Helsingfors are given in *Monthly Notices*, lxix. p. 392. The comparisons of the scales with the C. and M. scale are given below.

Value of the Potsdam and Toulouse Scales.

Potsdam or Toulouse Mag.	8·00	8·50	9·00	9·50	10·00	10·50	11·00	11·50	12·00
C. and M., Potsdam	8·07	8·68	9·35	9·91	10·41	10·80	11·16		
C. and M., Toulouse	8·13	8·35	8·84	9·21	9·71	10·05	10·64	11·14	11·52

Value of the Helsingfors Scale.

Helsingfors Mag.	9·00	9·50	10·00	10·50	11·00
C. and M. Mag.	9·07	9·71	10·13	10·51	10·87

Nizamiah Observatory, Hyderabad:
1915 *May* 27.

Methods of determining the Tilt of Photographic Plates.

By R. J. Pocock, B.A., B.Sc.

Last year, when erecting the Hyderabad astrographic telescope, the question of adjusting the tilt of the plates presented some difficulty. The photographic end of the telescope was made in the Observatory workshop, and so arranged that the plate rests on three steel screws, which I had lacquered to prevent rusting.

To prevent accidental movement of these screws, they are kept firm by three lateral tightening screws. The usual method of adjusting these screws is to use a small collimator attached to the back of a plate. This method was used here, fine cross-wires being stretched across the object-glass without touching the lens, and the screws adjusted so that these cross-wires, the centre of the plate, and the cross-wires in the collimator were in a line for all orientations of the collimator. The collimator used is a rather cumbrous instrument constructed in the workshop, and consequently this method did not seem altogether satisfactory—beyond assuring that the tilt was small. One objection to this method which may have escaped notice elsewhere is the possibility that the plate is not uniformly thick. Since the collimator is attached to the back of the plate, any such want of uniformity will introduce an apparent tilt where none really exists.

When testing the Oxford astrographic telescope for tilt about a year and a half ago, a thin plate (plate-glass) was used, and on measuring with calipers at the points where it was supported in the telescope the differences in thickness were found to be such as to produce an apparent tilt about equal to that usually found in such instruments.*

The adjustment of the Hyderabad telescope not being quite satisfactorily made, it seemed desirable, if possible, to determine the tilt from the measures themselves.

Tilt determined from Star Counts.

In the first place, the simple method of counting suggests itself. The star density varies in different parts of the plate: in the absence of any other cause for variation the density should be symmetrical about the plate centre—this will not be the case if the plate is tilted.

Let r_0 be the distance from the plate centre of the position of best focus for an untilted plate, r_1, r_2 the distances measured along (say) the x-axis for a tilted plate with the same centre. Let f be the focal length of the telescope, k the distance between the plate centre and the object-glass, ψ the tilt of the plate. Then we have the following geometrical relations:—

$$f^2 = r_0^2 + k^2$$

$$r_0^2 = r_1^2 - 2r_1 \,.\, k \sin\psi = r_2^2 + 2r_2 \,.\, k \sin\psi$$

$$r_1 - r_2 = 2k \sin\psi$$

$$r_0^2 = r_1 \,.\, r_2.$$

* See note, p. 665.

The perpendicular from the centre of the object-glass on the tilted plate meets it at a point distant $\frac{r_1 - r_2}{2} = k \cdot \sin\psi$ from the centre.

Professor Turner has given an approximate formula * for the density at any point of a plate, viz.

$$D = a\{1 \sim c(r^2 - r_0^2)\},$$

where a, c are certain constants. This formula agrees well with observation for all but large values of r.

Consider a circle of radius 13 réseaux intervals ($= b$ say) with centre at the plate centre: we may neglect the corners of the plate beyond this circle, as they contain few stars. We can integrate the above formula over a quadrant of this circle, and thus find the density D on one half the plate. This is on the assumption that for practical purposes we may replace one eccentric circle of best focus by two concentric semicircles.

$$D_1 = 2\int_0^{\pi/2}\int_0^b a[1 \sim c(r^2 - r_0^2)]r \cdot dr \cdot d\theta$$

$$= \pi a \left\{ \frac{b^2}{2} + \frac{cb^4}{4} - \frac{bcr_1^2}{2} + \frac{cr_1^4}{2} \right\}.$$

If we assume $r_1 = 30'$, $r_2 = 35'$ with the values $b = 65'$, $c = 10^{-4}$, we find

$$\frac{D_1 - D_2}{D_1} = \cdot 013.$$

In this case $\frac{r_2 - r_1}{2} = k \sin\psi = 2\frac{1}{2}' = 0{\cdot}5$ réseau intervals.

For a telescope of focal length $3^m{\cdot}44$ adjusted so that the circle of best focus has a radius of about $32'$, an excess of stars amounting to 1·3 per cent. may be due to a tilt of about $2'$.

If we could be sure that the star density was uniform over the region covered by the plate, this method would form a very delicate test of the tilt of a plate.

Application to the Oxford Plates.

Detailed counts of some Oxford plates are given in *Monthly Notices*, March 1908, p. 399.

If we consider the regions a to g of Table II. of that paper, these are practically the same as the circle radius 13 considered above.

Denoting the sum of XY + Xy for the regions a to g by X, etc., we find

$$X = 2212, \quad Y = 2313, \quad x = 2135, \quad y = 2034$$
$$X - x = 77, \quad Y - y = 279.$$

* *M.N., R.A.S.*, lxii. p. 434.

The plates are nearly all in the first three hours of R.A., and there is no sensible systematic increase in the number of stars with R.A.; we may therefore consider the difference $X - x$ as entirely due to instrumental causes. On the other hand, the difference $Y - y$ is probably in part at any rate due to a systematic difference in the number of stars depending on the declination. The different numbers of stars in the different volumes of the catalogue might furnish some indication of the magnitude of this difference, but it would hardly be reliable, in view of the great differences in sensitiveness of plates and in the amount of revision work done for some volumes. We shall consider, therefore, only the difference $X - x$.

We have

$$\frac{D_1 - D_2}{D_1} = \frac{77}{2212} = \cdot 035.$$

According to the above formula this would give a tilt of 5′ or 6′, not an improbable amount. The excess, of course, may be due to a combination of tilt of object-glass and tilt of plate.

The Hyderabad Plates.

The Hyderabad instrument, which is fitted with an 8-inch Cooke photovisual lens, does not appear to be well focussed. The instrument was focussed in 1914 December. It appeared then, from counts of stars on several plates taken for the purpose, that the density was very uniform over the plate: the position of best focus appeared to be near the regions *f*, *g*, as in the case of the Oxford plates. Since then, however, there seems to have been a considerable change in the focus, which may be partly due to the great difference in temperature between December and April. The position of best focus now appears to be the centre of the plate, and the density falls off rapidly towards the edge.

The following table represents the density in different regions of the plate:—

	a.	*b.*	*c.*	*d.*	*e.*	*f.*	*g.*	*h.*	*i.*	*k.*
xy	27	20	21	19	20	20	16	15	15	14
Xy	24	25	20	19	21	19	17	15	12	12
*x*Y	18	24	21	23	21	20	19	18	18	15
XY	24	22	21	21	21	20	21	17	15	13
Total	93	91	83	82	83	79	73	65	60	54
1914 Dec.	51	53	67	49	53	59	56	48	48	

It is possible that as at present adjusted the plate does not cut the surface of best focus at all, in which case the above formula may not be applicable; but it may be noted that, except for bright stars near the edge of the plate, the images do not appear sensibly out of focus. The numbers in the line 1914 December represent the density for intermediate regions at that date, but depend on fewer stars than the numbers above. The results for 1914 December

depend on counts of images on the plates; the later results depend on counts of actual measures and may be affected by a natural tendency to measure fewer stars near the edges of the plates where the images are bad—and in the case of the Hyderabad plates where the réseau lines are not quite so good. Colour is lent to this idea by the fact that the plate constants show no evidence of sensible change of focus. Taking the mean values of A, E for plates taken last December for the plates considered above and for still later plates, we find

A	−·01762,	−·01762,	−·01756
E	−·01777,	−·01784,	−·01768

The difference A − E is due to refraction; the mean refraction for plates on the meridian gives A − E = +·00014 in Zone −17°. We have

$$X = 4694, \quad Y = 4879, \quad x = 4661, \quad y = 4476,$$
$$X - x = 33, \quad Y - y = 403.$$

We can estimate the systematic difference $X - x$ from the plates themselves: thus, the first seven plates average 1000 stars each, the remaining eight average 700. The average increase between two plates is therefore 40. The average difference between two half plates is therefore 10, the earlier half being the larger.

Reducing this number in the ratio of the number of stars from a to g to the total number of stars on the plates, we obtain

$$X - x = -\left(\frac{10 \times 9355}{12598}\right) \times 15 = -111.$$

Therefore the difference which may be due to tilt is $X - x = 144$

$$\frac{D_1 - D_2}{D_1} = \frac{144}{4694} = \cdot 031.$$

If in reality the best focus is about $r = 32'$, this would mean a tilt of about $5'$, equivalent to a displacement of the centre a little more than 1 réseau interval.

No attempt has been made to determine the tilt of the object-glass by means of a collimator. The lens is only an 8-inch, and with the large collimator we have it would be impossible to test it far from the centre, so that the result would have little value and the risk of damaging the lens would be too great.

The Plate Constants obtained on the Assumption of a Tilted Plate.

Finally, we may determine the tilt of a plate by computing the actual point at which the perpendicular from the object-glass meets the plate. We can do this if we include in the equations for solving the plate the terms

$$kx^2 + lxy \quad \text{and} \quad kxy + ly^2 \quad \text{in } x, y \text{ respectively}$$

where (k, l) are the required co-ordinates. Thus k is the same as $\frac{r_1 - r_2}{2}$ or $k \sin \psi$ in the previous notation.

Let a_1, a_2, a_3, a_4 denote the mean residuals in x for each quadrant of the plate. Then if the reference stars are symmetrically grouped we have the four equations:

$$\begin{aligned} 6{\cdot}5a + 6{\cdot}5b + c + 42{\cdot}25k + 42{\cdot}25l &= a_1 \\ 6{\cdot}5a - 6{\cdot}5b + c + 42{\cdot}25k - 42{\cdot}25l &= a_2 \\ -6{\cdot}5a + 6{\cdot}5b + c + 42{\cdot}25k - 42{\cdot}25l &= a_3 \\ -6{\cdot}5a - 6{\cdot}5b + c + 42{\cdot}25k + 42{\cdot}25l &= a_4 \end{aligned}$$

and four similar equations in which d, e, f, l, k, β replace a, b, c, k, l, α respectively. From the equations we obtain

$$\begin{aligned} 169 \,.\, l &= (a_4 - a_3) - (a_2 - a_1) \\ 169 \,.\, k &= (\beta_4 - \beta_3) - (\beta_2 - \beta_1). \end{aligned}$$

If the plate is large the stars will be approximately symmetrically distributed, and the above coefficients 6·5, 42·25 will be nearly correct.

This method has been applied to some of the large Hyderabad plates, counts of which have been discussed above. The results are as follows:—

Plate No.	R.A. h m	k.	l.
405	7 0	+ ·000007	− ·000009
424	7 8	− ·000010	+ ·000002
402	7 16	− ·000005	+ ·000014
426	8 20	+ ·000012	+ ·000023

It is clear from these numbers that the measures are not affected by any appreciable tilt. If there is any tilt it is peculiar to each plate and may be treated as a small accidental error.

If the above numbers are correct, the Hyderabad plates may be considered free from any systematic tilt.

Nizamiah Observatory, Hyderabad:
1915 *April* 22.

[*Note added June* 20.—If the plate used for testing the tilt with a collimator is not of uniform thickness, but is placed in the four different orientations in the telescope, the collimator remaining perpendicular to the back of the plate, the tilt will be correct, provided the centre of the O.G. appears to describe a circle about the cross-wires of the collimator as centre.]

Measures of Double Stars. By Eric Doolittle.

The double star measures of the Flower Astronomical Observatory which were principally made during the years 1907–1911 were published two months ago by the University of Pennsylvania, as vol. iv. part 1 of their Astronomical Series; and this volume contained measures on some 1954 different pairs. The following paper contains the mean results of the measures on 416 additional pairs, omitted from this volume. It will be noticed that a large proportion of these are wide and neglected H pairs, on which measures have been made in the course of the work of bringing the General Catalogue of Burnham up to date. Those which were not identified in the G.C. have been either so identified in Argelander's Durchmusterung, or, when too faint for this, their position has been determined by measurement from the nearest DM star. These identifications have been here added in the Notes, but for brevity the resulting right ascensions and declinations are omitted. Many hundreds of measures on the 8000 new stars, discovered since the publication of the G.C., are reserved for a later publication.

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.		Notes.
			°	″					
9893	H 1475	1907·54	273·1	7·09	10·4	11·5	3		1
9895	H 1478	1907·54	34·9	2·27	9·5	10·1	3		2
9895*a*	Anon	1907·54	222·2	15·69	9·4	9·5	2		3
9898	H 1473	1907·54	156·1	19·28	9·2	9·9	3		4
9899	H 902	1907·45	197·6	6·55	8·9	9·1	3		
9901	H 903	1907·50	353·3	10·95	10·3	11·2	3		5
9903	H 1479	1907·54	1·5	33·89	7·7	9·8	3		
9904	H 1476	1907·54	71·9	12·95	9·3	10·6	3		6
9911	H 904	1907·47	317·8	27·76	8·5	9·3	3		
9917	H 905	1907·47	171·3	11·65	8·8	10·5	3		
9918	H 2931	1907·65	313·1	6·50	10·1	10·8	3	AB	7
		1907·69	209·8	14·08	...	12·7	2	AC	
9918*a*	Anon	1907·54	266·4	8·66	10·9	11·8	3		8
9920	H 2930	1909·52	156·4	28·37	10·7	11·2	2		9
9922	HCW 18	1905·79	314·8	21·21	8·0	8·7	2		
9925	Σ 2631	1907·02	339·6	4·65	...	...	3		
9927	A 278	1907·53	297·1	1·59	8·8	12·3	4		
9928	Hn 37	1911·52	308·8	3·10	9·0	11·3	3		
9930	Hu 80	1908·70	4·1	2·47	8·4	10·9	3		
9941	AG 247	1905·89	...	...	...	...	3	AB	10
		1905·95	27·9	36·01	8·9	11·9	2	AC	
9942	Dawes 12	1907·56	100·5	28·26	9·5	11·9	3		11

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.	
			°	″					
9943	A 382	1907·47	83·4	1·46	7·6	10·2	3		
9948	H 2933	1908·51	26·6	21·75	9·5	10·2	2		
9964	A 384	1908·52	357·7	1·09	9·0	10·8	3		
9967	H 1486	1908·52	45·9	6·13	10·0	10·4	2		12
9969	H 906	1905·83	154·9	11·84	9·5	9·9	2	AB	
		1905·83	349·2	34·72	...	11·2	2	AC	
9970	Espin 87	1905·79	297·6	9·23	9·3	9·3	4		
9977	A 281	1907·52	171·6	3·93	8·8	9·5	3		
9979	OΣ 400	1905·74	358·9	0·33	7·5	7·9	5		
9990	Doo 14	1909·91	260·3	1·72	9·2	9·9	3		
10015	H 2941	1910·51	109·7	6·24	9·6	10·0	2		
10018	H 1494	1910·51	0·9	9·10	10·0	11·7	2		13
10019	Hn 157	1906·52	236·7	2·27	9·6	10·2	2		
10026	Hd Zns	1908·44	257·8	2·04	8·9	11·2	3	AB	
		1908·45	270·7	32·01	...	10·0	2	AC	
10030	A 385	1908·76	257·3	2·58	9·8	9·8	3		
10043	H 2943	1910·56	97·7	8·10	9·9	11·0	2	AB	14
		1910·56	11·3	25·54	...	11·5	2	AC	
10046	Hu 268	1909·49	31·3	2·79	9·5	9·6	3		
10050	Ho 123	1909·50	225·1	2·96	9·2	9·2	3		
10052*a*	Anon	1909·51	242·4	13·56	9·3	10·5	2		15
10052*b*	Anon	1909·51	63·9	11·63	9·1	9·9	2		16
10053	A 389	1910·55	213·1	1·54	9·2	9·7	3		
10061	AG 251	1909·17	184·0	7·17	8·5	9·8	6		
10065	H 911	1909·49	124·1	20·12	9·1	9·9	2		
10072	β 984	1909·47	214·9	0·93	8·4	8·9	3		
10075	Hu 585	1910·75	48·4	4·71	8·5	9·7	2		
10076	β 441	1909·46	67·4	5·95	7·2	11·7	3		
10080	Hu 357	1909·21	197·0	1·92	7·7	11·2	3		
10086	H 5188	1905·80	60·9	4·66	7·2	10·0	3	AB	
		1905·80	321·4	27·44	...	8·3	3	AC	
10088	H 1498	1909·50	214·8	14·22	10·5	10·8	2		17
10094	AG 252	1910·56	126·3	10·43	8·9	9·0	3		
10103	OΣ 405	1905·68	151·8	0·73	7·4	9·5	3		
10114	H 2950	1909·51	288·2	13·22	9·4	10·0	2		
10119	H 1503	1908·76	74·4	15·66	9·1	9·7	2		
10134	β 431	1905·73	39·4	0·72	8·4	8·4	3		
10137	H 2954	1908·84	305·5	2·99	9·1	9·4	2		
10140	Σ 2668	1906·53	285·5	3·37	...	...	3		

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.	
			°	″					
10147	Σ 2672	1910·79	300·3	0·96	...	...	3		
10148	β 1259	1905·73	171·6	0·68	8·2	9·2	3		
10171	H 1510	1908·76	149·2	4·58	8·4	8·8	2	AB	
		1908·76	340·1	6·03	...	12·5	2	AC	
10177	H 1511	1908·76	239·3	10·30	10·2	11·0	2		18
10179	A 289	1908·76	153·5	3·42	8·2	11·8	3		
10182	Σ 2679	1908·52	78·2	22·94	...	...	2		
10184	H 1507	1910·51	64·2	10·01	10·0	10·8	2		19
10186	H 1509	1908·78	180·7	19·00	8·9	9·2	2		
10190	Howe 54	1906·51	52·9	2·37	8·0	8·7	2		
10191	Σ 2678	1906·52	314·8	3·41	...	...	3		
10192	Ho 457	1911·51	60·2	1·98	8·3	8·4	3		
10194	β 1134	1908·82	79·8	4·25	6·0	12·5	3		
10206	A 392	1909·48	302·1	0·83	9·2	11·5	3		
10211	A 47	1909·57	174·0	1·36	9·5	10·3	3		
10213	Dem 22	1910·75	144·5	2·85	...	...	3		
10220	Ho 130	1909·69	285·5	2·15	8·6	9·2	2		
10222	AG 255	1909·57	288·8	5·19	9·3	9·5	2		
10224	H 2966	1910·56	258·5	3·39	10·8	12·3	3		20
10231	AG 256	1910·55	352·3	4·86	8·8	9·2	2		21
10236	H 918	1910·43	248·2	3·55	10·8	11·5	2		22
10253	H 2970	1910·45	181·5	11·84	9·2	9·8	2		
10253*a*	Anon	1910·48	207·8	7·24	10·2	10·5	2		23
10260	H 1519	1909·70	227·7	26·72	10·7	12·0	2		24
10266	β 63	1911·51	345·8	1·08	6·2	8·5	3		
10282	H 1525	1909·51	232·9	8·36	8·5	8·8	2		25
10288	H 2974	1909·57	292·4	13·88	9·0	9·0	2		
10296	A 735	1910·48	272·0	4·55	8·9	10·7	3		
10297	Hn 161	1905·81	51·0	2·00	8·8	10·7	2		
10323	H 1540	1905·84	345·4	22·62	8·6	11·7	3	AB	
		1905·84	130·9	41·79	...	12·4	3	AC	
10362	Σ 2701	1906·52	221·5	2·16	...	...	3		
10363	β 151	1903·41	24·3	0·50	...	...	1		
		1906·59	89·5	0·30	...	...	1		
		1909·55	174·8	0·27	...	...	3		
10374	H 920	1910·52	211·7	13·74	9·6	10·7	2		
10379	H 1550	1910·54	223·4	6·89	9·6	10·7	2		26
10383	Hn 39	1907·57	176·2	7·76	8·5	10·7	3		
10384	A 744	1909·59	270·1	0·52	8·7	9·2	3		

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
				″				
10388	AG 261	1909·54	155°·6	3·90	9·0	9·7	2	
10390	OΣ 533	1906·16	302·0	16·96	...	11·0	4	
10397	Σ 2707	1905·83	31·4	22·83	7·5	8·5	3	AB
		1905·83	195·4	55·81	...	7·7	3	AC
10400	Hu 362	1909·58	310·9	0·49	9·1	9·8	3	
10417	HN 101	1905·81	324·1	23·58	8·4	8·8	3	
10418	Σ 2706	1905·77	37·8	11·32	8·1	9·9	2	
10422	Σ 2713	1906·79	243·5	5·07	8·7	8·9	3	
10425	Σ 2714	1905·93	338·6	6·60	...	...	5	
10448	AG 262	1910·50	273·0	4·86	9·5	9·5	2	
10455	Hn 162	1905·79	143·4	1·83	9·2	9·6	3	
10461	Σ 2720	1905·78	182·5	3·99	8·4	8·7	3	
10465	Σ 2721	1906·12	30·0	2·37	...	...	3	
10467	Σ 2722	1906·11	307·7	7·08	...	...	3	
10471	H 924	1910·45	84·2	7·72	9·8	10·5	2	27
10474	H 2992	1905·78	140·2	3·06	8·9	9·7	3	28
10478	Har 160	1910·51	202·1	5·62	9·6	10·3	2	29
10479	H 925	1910·48	173·6	9·75	8·9	9·7	2	
10480	A 171	1907·85	325·7	4·62	8·6	11·8	3	
10483	Σ 2723	1906·53	100·2	1·47	...	...	3	
10489	A 172	1906·52	221·8	2·77	9·0	10·2	3	
10501	AG 264	1910·55	356·8	1·85	8·8	8·9	3	
10519	β 364	1905·73	225·5	1·52	8·5	8·9	3	
10533	OΣ 413	1905·70	62·9	0·77	...	...	3	
10538	β 66	1905·73	159·1	1·21	8·5	9·0	3	
10559	Σ 2729	1903·74	276·4	0·20	...	...	1	
		1905·41	279·4	0·28	...	...	1	
		1909·52	320·7	0·2	...	...	1	
10562	Σ 2730	1911·50	334·4	3·44	...	...	3	
10568	A 612	1910·43	10·8	1·38	9·5	10·2	2	
10572	H 3003	1905·75	216·4	2·46	6·7	8·5	6	
10573	HC Wil 21	1905·72	18·7	30·69	8·8	9·4	4	
10581	Ho 144	1909·59	170·7	0·36	7·2	7·2	2	
10584	H 1585	1910·49	193·5	13·17	8·9	11·5	2	
10587	Ho 145	1908·79	310·1	9·11	8·5	10·0	3	
10607	β 367	1905·69	151·9	0·36	7·5	7·5	1	
		1910·46	161·9	0·32	...	...	2	
10627	Hd Zns	1908·45	333·3	11·16	9·0	13·2	2	AB
		1908·45	137·9	41·47	...	9·4	2	AC
10680	Hd Zns	1908·44	...	...	...	...	2	30

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
10685	Σ 2744	1911·51	157·1	1·44	...	...	3	
10690	Σ 2746	1906·55	295·4	1·10	...	...	3	
10692	Leav 9	1906·54	196·1	1·83	9·2	11·2	2	
10696	β 156	1905·74	243·3	1·15	7·4	9·3	3	
10704	H 3007	1905·72	224·2	6·81	8·5	8·8	4	
10714	β 70	1906·55	95·1	5·31	10·7	11·1	3	
10716	H 1608	1910·51	257·5	19·85	7·0	11·0	2	
10730	Hall	1910·46	...	...	...	...	6	31
10774	Σ 2771	1905·95	211·6	2·46	9·0	9·0	2	
10775	Σ 2763	1906·75	292·9	17·10	...	...	2	
10778	Har 163	1910·75	...	...	...	...	2	32
10779	Hd Zns	1908·51	317·8	0·82	8·7	8·9	3	
10783	H 5251	1910·67	304·5	8·33	8·1	8·5	2	
10784	Duner 3	1909·73	197·2	6·29	9·3	9·5	3	
10786	H 1616	1909·69	278·6	7·61	10·7	11·7	2	33
10788	Ho 151	1911·50	11·3	1·25	8·4	8·8	3	
10794	Σ 2765	1911·50	81·9	2·95	...	...	3	
10799	Σ 2768	1910·67	195·2	7·41	7·0	9·2	3	
10809	OΣ 430	1909·59	210·3	1·49	...	...	3	
10832	Σ 2780	1905·95	226·2	1·36	6·0	7·4	2	
10836	H 1626	1910·72	166·7	8·81	9·3	10·5	2	
10847	H 1628	1908·79	253·7	15·25	9·5	9·9	2	
10851	β 1261	1905·75	150·9	1·54	8·3	10·7	3	
10896	Σ 2786	1911·54	184·0	2·54	...	...	2	
10901	AG 270	1910·70	112·9	5·72	8·5	8·8	2	
10914	OΣ 435	1911·53	36·5	0·73	7·7	8·2	3	
10915	A 295	1907·88	226·3	0·46	8·1	8·2	3	
10919	β 1262	1908·71	112·2	1·91	8·0	9·0	2	
10922	OΣ 437	1905·71	40·3	1·64	6·2	6·8	4	
10930	Espin 137	1905·97	74·0	45·87	6·7	9·3	2	AB
		1905·97	70·3	2·67	...	12·3	2	BC
10941	Espin 138	1906·03	264·7	9·59	6·8	12·7	4	
10946	Σ 2791	1910·67	104·6	2·53	...	...	3	
10948	Σ 2792	1906·13	330·7	6·74	...	...	3	
10950	Hu 369	1910·72	10·7	1·34	9·0	13·0	2	
10955	Σ 2795	1905·94	301·4	1·63	8·4	9·3	2	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
10961	H 3027	1905·94	122·7	33·88	9·0	9·0	2	
10982	H 3028	1910·55	242·8	20·51	9·5	10·8	2	
10986	Σ 2798	1906·68	146·1	6·71	8·0	9·7	3	
11000	Kust 61	1909·86	271·9	3·96	9·5	9·9	3	
11001	Σ 2799	1910·57	295·8	1·43	...	...	3	
11010	Σ 2807	1906·05	315·1	2·01	8·3	8·7	4	
11028	Hn 45	1909·63	199·5	1·35	8·6	9·4	3	34
11029	H 3031	1910·54	253·6	26·70	8·8	10·2	2	
11037	H 3033	1909·70	244·7	24·57	8·4	9·3	2	
11041	H 1655	1909·70	28·7	20·39	8·8	10·8	2	
11051	Σ 2804	1908·88	336·8	2·76	...	...	3	
11061	Σ 3112	1906·95	239·0	7·31	...	...	4	
11068	β 74	1911·50	321·2	1·49	7·2	9·0	2	
11102	Hn 46	1909·63	198·1	1·24	9·9	10·3	3	
11104	H 1667	1910·57	199·8	14·74	10·3	11·3	2	
11105	H 1668	1910·57	35·5	8·30	8·6	9·8	2	
11115	H 941	1910·80	334·7	27·21	6·0	12·2	2	
11122	H 1670	1910·34	261·2	11·55	10·1	10·4	2	35
11125	β 1212	1909·85	281·1	0·58	6·3	7·2	3	
11127	Cord Zns	1908·74	65·5	4·86	8·0	9·0	3	
11130	OΣ 445	1905·71	113·1	0·86	8·1	8·8	3	
11137	AG 274	1909·50	153·1	9·00	8·8	9·5	2	
11150	H 1680	1906·06	263·9	15·75	9·7	12·0	2	AB
		1906·06	218·2	23·17	...	12·5	2	AC 36
11168	OΣ 448	1906·54	231·4	0·81	...	...	2	
11174	Hu 279	1910·51	356·2	2·26	9·2	9·6	3	
11193	H 1683	1909·51	169·9	12·32	9·0	9·8	2	AB
		1909·51	311·0	25·52	...	9·6	2	AC
11200	β 688	1905·68	200·4	0·48	7·4	7·5	3	
11210	Ho 166	1910·79	68·3	0·39	7·4	7·4	4	
11214	Σ 2822	1909·80	125·3	2·26	...	...	3	
11221	β 1305	1909·84	39·9	0·75	9·5	11·2	2	
11222	β 989	1910·79	291·1	0·20	...	...	2	
11233	AG 276	1910·52	356·7	2·07	9·4	9·9	2	
11240	AG 277	1910·52	54·5	2·35	8·8	9·0	3	
11246	Σ 2825	1909·72	116·6	1·01	...	...	3	
11256	Bar 15	1910·47	78·7	0·42	8·8	9·3	3	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
11257	H 943	1909·79	142·8	5·58	10·5	11·5	2	37
11263	Hu 378	1908·82	344·9	0·72	8·8	9 7	3	
11272	Σ 2828	1909·88	141·1	28·08	...	...	2	
11273	H 3060	1909·75	212·6	15·38	9·4	11·2	2	
11280	H 286	1909·75	260·5	21·07	8·8	12·0	2	
11307	Hn 49	1909·63	342·7	2·28	8·7	11·3	3	
11329	β 1213	1906·81	308·5	0·83	9·2	10·8	2	
11331	Krue 56	1905·86	303·4	4·14	9·0	9·4	2	
11346	β 75	1911·54	44·7	1·13	8·1	8·6	3	
11358	AG 278	1909·51	158·5	3·22	9·0	9·5	2	
11374	H 1706	1910·55	283·1	8·86	9·0	10·0	2	38
11375	H 1707	1910·56	331·4	8·37	8·8	10·0	2	39
11381	H 1708	1910·57	236·5	7·14	9·6	10·2	2	40
11392	H 3075	1905·72	306·0	7·03	10·3	10·9	4	
11394	H 3076	1905·72	249·6	34·19	8·0	10·1	4	
11401	H 3078	1910·55	189·4	4·35	9·6	10·2	2	
11407	Espin 145	1906·04	206·0	2·76	9·1	10·0	2	
11408	A 305	1907·89	224·3	2·74	8·7	11·2	3	
11459	β 694	1905·68	354·2	0·64	6·0	8·8	3	
11472	OΣ 460	1907·79	347·5	8·81	7·7	11·0	3	AB
		1907·79	41·0	16·84	...	11·7	3	AC
11476	Har 169	1910·53	31·3	2·26	8·5	9·0	2	41
11486	Σ 2861	1906·13	221·3	6·98	...	...	3	
11492	A 183	1907·45	240·2	0·77	9·1	9·9	3	
11493	A 407	1907·45	22·2	0·63	9·2	9·2	3	
11513	H 1731	1907·67	207·2	13·74	8·6	10·7	3	
11513*a*	Anon	1907·60	220·4	4·72	9·5	9·7	3	42
11515	Hu 284	1906·68	111·7	3·34	8·6	9·0	3	43
11523	Σ 2868	1906·69	359·8	1·12	...	...	4	
11534	H 956	1909·70	303·1	10·93	9·6	9·8	2	
11535	H 1737	1907·68	165·6	6·46	8·8	8·8	3	
11541	H 1738	1907·68	178·6	7·90	9·6	11·0	3	44
11556	OΣ 464	1908·07	65·2	0·65	...	...	4	
11567	Kust 63	1907·55	241·2	4·07	9·2	9·9	3	
11570	A 409	1907·47	25·2	0·44	9·3	9·4	4	
11573	H 958	1911·54	230·5	5·42	9·5	10·4	2	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
11588	Ho 291	1908·84	199·2	9·48	7·0	12·2	2	
11590	Σ 2878	1908·71	122·8	1·29	...	...	2	
11596	Har 170	1908·70	241·8	7·27	9·5	9·9	3	
11599	Σ 2881	1910·75	99·0	1·65	...	...	3	
11601	AG 281	1908·84	21·2	2·32	9·0	10·0	2	
11616	H 960	1907·95	262·3	6·73	9·4	10·7	3	45
11639	H 3105	1909·70	116·9	19·28	8·7	11·5	2	
11657	Σ 2895	1911·82	35·9	8·40	...	...	3	
11657*a*	*β* —	1911·82	78·6	12·43	11·0	11·7	2	46
11666	Ho 615	1909·82	17·8	2·42	9·6	10·8	3	
11674	OΣ(App)231	1908·73	110·2	91·27	...	...	2	
11681	Kust 64	1909·69	114·4	6·15	10·5	11·2	2	
11687	Ho 182	1911·93	133·1	1·78	8·6	8·7	2	
11688	H 3109	1908·73	302·4	28·01	9·2	10·0	2	
11691	*β* 172	1909·84	358·6	0·82	6·5	6·5	3	
11701	Weisse 38	1907·56	52·8	6·61	9·0	9·4	3	
11703	H 1706	1907·95	336·8	6·27	9·3	11·3	3	47
11706	H 1763	1907·95	111·8	9·74	10·7	11·0	3	48
11711	H 963	1909·80	60·5	7·13	9·4	10·2	2	
11736	*β* 701	1905·02	262·2	1·28	7·2	11·0	4	
11737	Σ 2908	1906·78	113·8	9·19	...	...	2	
11741	Σ 2910	1908·79	339·4	5·44	...	...	3	
11747	Hd Zns	1908 70	181·5	3·13	9·4	9·7	3	
11757	H 1772	1908·78	104·9	6·39	10·6	10·8	2	
11761	Krue 60	1906·98	111·7	3·37	...	...	2	
		1907·21	106·2	3·30	...	...	2	
		1908·83	103·3	3·24	...	...	5	
		1909·75	99·3	3·00	...	...	3	
11789	Σ 2915	1906·69	147·3	12·89	...	...	3	
11793	Ho 475	1909·79	321·4	0·81	8·5	9·0	3	
11796	Σ 2919	1906·69	261·0	17·31	...	...	3	
11798	AG 283	1910·81	336·7	2·78	9·0	9·3	3	
11806	H 1781	1909·79	289·3	21·36	9·3	12·5	2	
11823	Hn 51	1906·78	181·3	1·12	8·4	8·9	3	
11825	H 966	1909·80	268·9	12·91	7·3	11·5	2	
11828	Σ 2924	1906·45	274·6	0·75	...	...	3	
11849	Hu 391	1909·97	169·7	0·88	9·2	10·0	3	
11861	Ho 480	1909·76	226·3	0·81	8·2	9·7	2	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
11865	Hu 392	1909·85	342·1	0·55	9·1	9·1	3	
11866	Σ 2928	1908·14	311·0	4·00	...	...	4	
11897	Σ 3134	1906·95	75·9	6·38	...	...	4	
11930	OΣ 477	1905·71	208·1	5·05	7·2	11·7	3	
11952	Σ 2941	1911·53	263·3	11·37	...	...	3	
11962	OΣ 529	1905·71	201·7	3·53	8·8	9·8	2	AB
		1905·71	219·3	20·73	...	10·2	2	AC
11982	H 1811	1909·75	152·3	6·20	8·9	9·9	2	
12007	H 1819	1908·47	72·8	17·74	8·2	9·7	5	AB
		1909·12	208·8	5·46	...	12·3	3	BC
12026	H 972	1907·58	195·0	25·07	8·3	9·1	3	
12036	β 382	1909·34	241·8	0·88	5·7	8·2	4	
12054	Kust 66	1909·69	3·1	3·64	9·5	9·6	2	
12085	A 191	1907·68	225·7	2·37	9·3	11·3	3	
12090	OΣ 536	1906·80	168·0	0·33	...	...	2	
12094	OΣ 483	1910·80	235·5	1·06	...	...	3	
12095	Σ 2967	1906·81	6·3	6·85	...	...	3	
12101	Hn 55	1907·69	191·2	1·75	9·3	10·3	3	
12109	H 1839	1907·69	294·6	14·55	8·4	11·0	3	
12116	A 193	1907·53	177·3	1·40	9·0	9·6	3	
12133	AG 289	1911·53	226·0	1·65	9·2	9·6	3	
12139	A 195	1907·53	32·0	1·50	8·9	11·2	3	
12171	β 1025	1907·79	270·1	0·76	7·5	10·8	3	
12190	H 304	1908·77	169·9	21·18	8·9	9·8	2	
12225	Σ 2989	1910·68	143·9	1·65	...	...	3	
12235	A 199	1907·53	277·9	2·03	8·7	11·0	3	
12236	H 981	1908·77	281·2	17·72	8·6	10·2	2	49
12253	A 787	1910·87	197·6	2·75	9·2	9·2	3	
12257	β 1220	1909·13	105·3	0·50	8·9	9·4	3	
12276	β 79	1910·79	72·4	1·13	7·7	9·8	3	
12284	H 5393	1906·15	302·5	19·13	7·7	9·7	4	
12291	H 1866	1909·67	154·6	13·70	9·9	11·3	2	50
12293	A 202	1907·56	257·3	2·63	8·4	9·8	3	
12297	H 307	1909·67	313·5	25·33	9·3	10·5	2	51
12302	H 309	1909·67	326·6	14·04	11·0	12·0	2	52
12303	H 308	1909·67	309·9	17·87	9·9	11·5	2	53
12312	OΣ 494	1911·94	81·0	3·41	...	...	3	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
12317	Σ 3006	1907·64	166·0	5·54	8·2	8·9	4	
12325	β 718	1907·82	90·4	0·57	5·0	9·0	3	
12335	Ho 300	1910·50	...	...	...	...	3	54
12340	Σ 3008	1911·87	230·2	3·52	...	...	4	
12344	Σ 3009	1906·79	230·1	7·17	...	...	3	
12345	β 854	1910·57	88·3	2·32	...	...	3	
12347	H 3190	1910·56	0·6	17·60	10·2	11·7	2	55
12353	H 3192	1908·81	116·1	27·28	9·5	9·9	2	
12361	H 3193	1906·49	218·2	34·48	9·3	10·3	4	
12363	Ho 489	1910·79	243·5	0·56	...	...	3	
12381	Σ 3014	1911·74	280·3	8·19	8·0	10·7	3	
12382	H 1883	1908·93	145·7	18·47	9·3	9·6	2	56
12390	β 1222	1911·94	38·4	1·09	9·0	9·1	3	
12395	A 109	1907·53	312·9	0·67	9·0	9·5	3	
12432	β 720	1907·54	350·5	0·43	6·2	6·8	3	
12433	A 791	1907·61	350·3	0·84	9·8	9·8	3	
12444	H 3203	1910·68	209·1	6·55	8·9	9·3	2	
12459	Hn 58	1911·92	355·6	3·34	8·7	11·2	3	
12460	H 988	1909·86	243·4	8·72	9·8	11·2	2	
12466	H 315	1908·77	248·1	20·25	9·3	9·5	2	
12467	H 3208	1909·86	153·6	11·58	9·0	9·0	2	
12468	OΣ 500	1906·63	329·7	0·67	6·2	7·3	8	
12508	A 423	1910·67	168·2	1·57	9·0	10·8	3	
12515	Har Zns	1908·72	146·4	76·12	9·5	13·3	2	
12517	OΣ 503	1911·94	131·7	1·55	...	...	3	
12532	AGC 14	1907·53	203·8	1·47	6·3	9·3	3	
12541	Hu 300	1908·87	122·2	1·11	8·9	9·2	3	
12542	OΣ 305	1910·72	58·2	2·36	...	...	3	
12551	Egbert 8	1909·89	87·2	1·35	8·6	8·9	2	
12555	Bar 19	1909·91	169·1	0·64	8·4	8·6	3	
12573	OΣ 507	1907·28	267·5	0·68	6·0	7·2	6	
12601	Σ 3042	1906·68	87·5	5·18	...	...	3	
12606	Ho 204	1909·79	356·1	5·72	7·5	10·0	2	
12623	H —	1908·73	268·6	6·94	5·7	6·5	2	
12635	AG 296	1909·80	53·4	5·12	8·7	9·0	2	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
12636	A 425	1909·80	161·6	1·63	9·1	9·6	3	
12639	Σ 3046	1911·85	252·4	3·31	...	...	3	
12648	AG 297	1909·80	315·3	1·87	8·9	9·4	2	
12654	H 321	1909·76	133·1	21·07	7·0	11·2	2	
12657	H 995	1908·82	19·1	5·96	8·0	11·2	2	57
12675	Σ 3050	1910·52	219·7	2·27	...	...	3	
12687	β 732	1906·77	151·5	5·87	9·2	11·0	2	
12696	Hn 60	1910·78	301·9	0·26	...	...	2	
12697	Ho 208	1910·15	232·3	0·79	7·7	10·2	3	
12709	β 281	1907·90	198·9	1·56	7·2	9·8	3	
12723	A 429	1909·90	166·2	0·44	8·6	9·3	3	
12724	Σ 3055	1907·57	359·9	5·75	7·0	10·5	3	
12726	H 1931	1908·78	114·7	22·46	7·5	10·2	2	
12731	Σ 3056	1910·85	145·6	0·66	...	...	3	
12750	Σ 3060	1910·69	121·6	3·63	...	...	3	
12752	Hu 502	1908·77	109·8	2·59	7·8	11·3	3	
12943	Hu 1061	1910·83	47·4	2·81	8·6	12·5	3	
12950	Hu 1065	1910·76	342·3	3·25	9·5	12·3	3	
12954	A 989	1911·95	4·1	3·20	9·2	9·3	3	
12960	A 832	1910·83	111·4	1·74	8·9	10·7	3	
12992	Hu 1080	1910·51	256·8	0·42	7·0	7·5	3	
13152	Hu 849	1910·23	285·4	1·37	8·7	9·0	3	
13153	Hu 850	1910·21	359·8	0·71	8·5	9·5	3	
13154	Hu 1122	1910·24	168·1	2·73	9·0	10·2	3	
13156	Hu 1123	1910·51	159·9	0·50	8·3	8·8	3	
13164	Hu 856	1910·66	269·3	0·33	7·5	8·1	2	
13167	Hu 857	1911·22	222·3	4·36	8·6	11·2	3	
13168	Hu 858	1911·22	154·7	0·72	8·7	9·3	3	
13172	Hu 860	1910·52	19·1	0·35	9·3	9·5	3	
13180	Hu 867	1911·20	179·0	2·19	8·0	12·3	3	
13188	Hu 1127	1910·23	31·3	0·59	9·0	10·1	3	
13210	Hu 885	1910·21	290·0	2·24	8·8	10·8	3	
13216	Hu 887	1910·21	301·4	1·52	8·0	10·5	3	
13217	Hu 888	1910·21	150·3	0·70	8·0	8·6	3	
13222	Hu 890	1910·21	64·0	1·58	8·8	10·0	3	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
13226	Hu 1138	1910·21	2·3	1·17	9·5	10·2	3	
13228	A 1090	1910·32	85·9	1·84	8·8	11·0	3	
13230	Hu 892	1910·58	164·1	0·82	9·1	9·1	3	
13231	Hu 893	1910·23	32·7	1·66	9·5	9·9	2	
13234	Hu 894	1910·84	147·9	0·93	8·8	8·9	3	
13239	Hu 1143	1910·60	95·3	0·76	9·0	10·5	3	
13245	Hu 1146	1911·38	43·4	4·01	8·6	12·0	3	
13249	Hu 896	1911·38	12·5	1·50	8·6	10·2	3	
13251	Hu 897	1911·38	342·5	0·53	8·7	10·3	3	
13252	A 685	1911·41	8·4	0·72	8·2	11·0	3	
13260	A 1098	1911·16	234·4	4·77	8·8	10·5	2	
13261	Hu 899	1911·38	295·7	1·51	8·9	11·8	3	
13269	A 1103	1911·16	202·8	4·96	9·0	10·5	2	
13270	A 1104	1911·16	268·5	0·83	8·7	8·8	3	
13279	A 1109	1910·92	39·6	0·72	7·2	10·8	3	
13282	A 1110	1911·19	270·6	0·28	7·0	7·0	3	
13283	A 1111	1911·25	296·1	2·33	9·0	11·2	2	
13285	Hu 1153	1911·40	282·1	4·31	7·9	11·6	3	
13287	Hu 1154	1911·42	328·8	3·30	9·0	13·2	2	
13291	Hu 1155	1911·26	13·7	3·72	9·2	11·2	2	
13292	Hu 1156	1911·26	298·6	3·68	9·2	11·2	3	
13299	A 1116	1910·91	21·8	0·53	7·7	7·8	3	
13304	Hu 1160	1911·32	223·4	2·01	8·2	10·0	2	
13307	A 1119	1911·30	5·6	1·21	8·7	8·9	2	
13312	A 1122	1910·65	16·6	0·48	7·8	10·8	3	
13322	A 1125	1911·23	287·0	1·36	8·8	11·0	2	
13324	A 1126	1911·20	237·8	0·25	...	...	2	58
13326	A 1128	1911·25	347·5	1·53	8·9	9·7	2	
13329	A 1130	1911·26	334·8	2·00	9·0	10·7	2	
13369	A 1148	1911·34	354·7	2·02	8·8	10·3	3	
13394	A 1162	1911·38	164·3	1·72	8·5	10·7	2	
13398	A 1163	1911·33	104·5	0·90	8·2	10·5	2	
13406	Hu 925	1911·42	328·8	2·01	9·0	10·2	3	
13413	Hu 926	1911·46	267·0	3·71	9·0	9·1	3	
13430	Hu 935	1911·92	151·5	3·21	9·2	9·5	2	

G.C.	Star.	Date.	*p.*	*r.*	Mags.		Nights.	Notes.
			°	″				
13435	Hu 1192	1909·59	49·4	2·62	8·2	10·2	2	
13436	A 861	1911·40	341·9	1·61	8·8	10·5	2	
13437	A 862	1911·44	158·1	4·54	8·6	12·0	2	
13471	A 1184	1911·50	102·8	1·36	8·8	9·6	3	
13473	A 1185	1911·50	189·6	3·42	9·2	9·8	3	
13612	A 1228	1907·51	350·2	3·47	8·9	10·5	3	
13633	A 1235	1910·68	12·3	1·35	9·2	9·4	2	
13634	A 1236	1910·68	339·2	0·77	9·5	11·3	3	
13642	Hu 995	1910·69	187·9	1·44	8·9	10·3	3	
13646	A 1239	1910·70	52·1	2·18	9·0	10·7	3	
13657	A 1244	1907·70	272·3	2·41	9·5	10·7	3	

Notes.

(1) 200″ f and 1′ n of +40°, 4000.
(2) +43°, 3470. AC=89°: 8″·11·8; AD=310°: 9″·12·5.
(3) This may be H 1478; 7·5 s pr 9895 and 4′ s.
(4) Identified by β with +26°, 3785, but this star is not double. Near +26°, 3780.
(5) 28 s pr and 1′ s of H 904.
(6) Many other faint stars are near.
(7) 1 m 55·5 s f and 2′·5 s of +17°, 4209.
(8) 10 s f and 2′ n of +17°, 4233.
(9) Much fainter than was to be expected. 1 m 1 s f and 4′ n of +3°, 4254.
(10) The star is certainly single; 0″·3 is widely separated. The companion measured has, of course, nothing to do with the suspected duplicity.
(11) +27°,3633. This may, or may not, be the pair seen by Dawes. No other pair near.
(12) 1 m 48 s f and 3′ n of +10°, 4191.
(13) 1 m 58 s f and 4′ n of +11°, 4174.
(14) 32·5 s p and 4′ n of Alpha Capricorni. This also equals the Wilson pair measured in 1882. Two other similar pairs are in the immediate vicinity.
(15) −12°, 5676.
(16) 30 s p 3 Capricorni and 2′ s.
(17) 1 m 12 s p and 5′ n of +10°, 4250.
(18) 4·5 s f and 15″ s of +47°, 3089.
(19) This is also identical with 10185, H 1508. 14 s p and 7′ n of +14°, 4285.
(20) 50·5 s f and 5′ n of +7°, 4469.
(21) The measure of Leipzig is in error.
(22) 1 s f and 7′ n of −7°, 5308.
(23) 15 s f and 2′ s of +2°, 4180.
(24) 9 s p and 3′ n of +26°, 3912. Many other faint pairs. One was measured, 6 s p, which may be the H star.
(25) There is no star in the vicinity at all corresponding to O. Stone 50.
(26) +21°, 4299.
(27) −5°, 5361.
(28) No star here corresponding to Cinn. measures.
(29) −9°, 5570.
(30) Certainly no companion.
(31) All near-by stars examined; certainly nothing within 0″·25.
(32) Nothing here of any interest. A 60″ pair is nearest to the Hd place.
(33) 22 s p and 6′ n of +30°, 4342.
(34) This quadrant is certainly correct.
(35) 15 s f and 7′ n of +29°, 4664.
(36) Near +63°,1755.
(37) 10 s f and 7′ s of +25°, 4617.
(38) 56 s p and 5′ n of +28°, 4228.
(39) n p +31°, 4290.
(40) s f +22°, 4506.
(41) −2°, 5692.
(42) 1 m 51·5 s p and 30″ s of +40°, 4728.

(43) This is identical with Hu −, 11518.
(44) +45°, 3815.
(45) +29°, 4620.
(46) In field with 11657; as yet there is no change.
(47) +26°, 4423.
(48) 52·5 s p and 3′ n of +23°, 4532.
(49) +2°, 4636.
(50) 34 s f and 1′ s of +12°, 4966.
(51) +12°, 4971.
(52) 15 s f and 1′ s of H 307.
(53) 9 s f and 3′ s of H 307.
(54) Perfectly round; 0″·2 widely and easily separable.
(55) 13 s f and 6′ n of +5°, 5165.
(56) +45°, 4229.
(57) Decided change; probably the measures of Ho in 1893 belong to this.
(58) A 1110 is widely separable, but this is much closer, though A makes it wider. Increase in angle is indicated.

University of Pennsylvania:
1915 *March* 30.